NOTES FOR THE ASSOCIATION.

The Trustees of the Association have approved the establishment of a Michigan Section.

They have elected to membership in the Association the following one hundred applicants, on applications duly certified:

BEULAH M. ARMSTRONG, Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill.

W. S. BARLOW, Grad. U. S. N. A. Instr., Detroit Jr. Coll., Detroit, Mich.

M. A. BASOCO. Asst., So. Branch, Univ. of California, Los Angeles, Calif.

C. A. BENANDER, A.B. (Harvard). Instr., Penn. State Coll., State College, Pa.

H. A. BENDER, Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill.

J. A. BENNER, A.B. (Penn. State Coll.). Instr., Lafayette Coll., Easton, Pa.

JOHN BIGGERSTAFF. Student asst., Reed Coll., Portland, Ore.

P. N. BLESSING, A.B. (Swarthmore). Instr., Univ. of Michigan, Ann Arbor, Mich.

R. F. BORDEN, Ph.D. (Illinois). Instr., Brown Univ., Providence, R. I.

W. R. BOWERS, A.M. (Columbia). Prof., Radford State Normal Sch., East Radford, Va.

H. E. BRAY, Ph.D. (Rice). Instr., The Rice Inst., Houston, Tex.

C. T. BUMER, A.M. (Harvard). Instr., Ohio State Univ., Columbus, Ohio.

L. H. BUNYAN, B.S. (Wesleyan). Asst., Univ. of Wisconsin, Madison, Wis.

F. J. H. BURKETT, A.M. (Pittsburgh). Asst. prof., Trinity Coll., Hartford, Conn.

G. P. CAHOON, B.S. (Hamline). Prin., Parker Coll., Winnebago, Minn.

R. C. CARLSON, A.M. (Columbia). Head of dept. of math., Senior and junior high schools, New Brunswick, N. J.

PEARL E. CLARK, A.M. (Montana). Teacher, Chaffey Jr. Coll., Ontario, Calif.

W. M. COATES, A.B. (Williams). Instr., Univ. of Michigan, Ann Arbor, Mich.

C. J. COE, A.M. (Harvard). Asst. prof., Univ. of Michigan, Ann Arbor, Mich.

C. C. CRAIG, A.M. (Indiana). Instr., Univ. of Michigan, Ann Arbor, Mich.

ORPHA A. CULMER, A.M. (Montana; Michigan). Head of dept. of math., State Normal Sch., Florence, Ala.

OSCAR DAHLENE, M.S. (Kansas). Prof., Engg. and math., Univ. of Alabama, University, Ala.

J. W. R. H. DE LA, B.S. (S. D. State Coll.). Tutor, South Dak. State Coll., Minot, S. D.

O. C. EDWARDS, M.E. (Penn. State Coll.). Asst. prof. of mech. eng., Gen. ext. div., Univ. of Minnesota, Minneapolis, Minn.

E. F. FARNER, A.M. (Kansas). Head of dept. of math., High sch., Parsons, Kans.

C. E. FERGUSON, A.M. (Missouri). Head of dept. of math., Austin State Normal Coll., Nacogdoches, Tex.

C. B. FISHER, A.M. (West Virginia). Dept. of math., Potomac State Coll., Keyser, W. Va.

R. L. FLANDERS, B.S. (Norwich). Instr., Norwich Univ., Northfield, Vt.

W. McK. GAFAFER, A.M. (Columbia). Lecturer, Columbia Univ., New York, N. Y.

H. R. GARRETT, A.M. (Milligan Coll.). Prof., Lincoln Memorial Univ., Harrogate, Tenn.

E. R. GAY, A.B. (Harvard). Asst. dean, Harvard Coll., Cambridge, Mass.

WILLIAM GILLESPIE, Ph.D. (Chicago). Prof., Princeton Univ., Princeton, N. J.

A. G. HALL, Ph.D. (Leipzig). Prof. of math. and registrar, Univ. of Michigan, Ann Arbor, Mich.

BLANCHE HALL, A.M. (Illinois). Instr., Georgetown Coll., Georgetown, Ky.

G. P. HARMOUNT, A.M. (Ohio State). Head of dept. of math., East High Sch., Columbus, Ohio.

ISABEL HARRIS, A.M. (Columbia). Asso. prof., Westhampton Coll., University of Richmond, Va.

H. E. HARTIG, B.S. (Minnesota). Instr., Math. and mech., Univ. of Minnesota, Minneapolis, Minn.

R. E. HENRY, A.B. (Kentucky). Instr., High Sch., Madisonville, Ky.

A. O. HICKSON, A.B. (Acadia). Instr., Brown Univ., Providence, R. I.

GRACE H. HILL, A.B. (N. J. Coll. for Women). Clinton, N. J.

R. C. HUFFER, A.M. (Illinois). Instr., Michigan Agric. Coll., East Lansing, Mich.

G. H. HUNT, C.E. (Cornell). Instr., Applied math., So. Branch, Univ. of California, Los Angeles, Calif.

L. W. HUSSEY, A.B. (Dartmouth). Grad. scholar, Harvard Univ., 1923–1924.

G. S. JONES, B.S. in E.E. (Ohio Univ.). Athens, Ohio.

D. K. KAZARINOFF, Ancien élève (Moscow). Instr., Univ. of Michigan, Ann Arbor, Mich.

G. H. KEULEGAN, A.M. (Ohio State). Assoc. physicist, Bureau of Standards, Washington, D. C.

E. E. LIBMAN, Ph.D. (Illinois). Instr., Univ. of Illinois, Urbana, Ill.

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HUGH MCGILVRAY, A.B. (Oklahoma). Norman, Okla.

MARGARET H. MCGUIRE, B.S. (Minnesota). Instr., Univ. high sch., Univ. of Minnesota, Minneapolis, Minn.

J. R. McKee. Instructing federal students, College Station, Tex.
Mary K. Mickle, A.B. (Alabama). Roanoke, Ala.
Aaron Miller, A.B. (Indiana). Prin., Geneva high sch., Flat Rock, Ind.
J. E. Minister, B.S. in E.E. (Ohio Univ.). Chauncey, Ohio.
E. E. Moots, M.S. (Maine); C.E. (Wisconsin). Head of dept. of math. and eng., Cornell Coll., Mt. Vernon, Iowa.
Selah W. Mullen, B.S. (Michigan). Dept. of math., North Eastern High Sch., Detroit, Mich.
Ada A. Norton, Ph.B. (Albion). Asso. prof., Michigan State Normal Coll., Ypsilanti, Mich.
Frank Ordway, B.S. (Alabama Polytechnic). Head of dept. of math., Central High Sch., Birmingham, Ala.
J. N. Peck, B.S. (Fremont). Prof., Berea Coll. Acad., Berea, Ky.
F. W. Perkins, Jr., A.M. (Harvard). Grad. student, Harvard Univ., Cambridge, Mass.
W. J. Pickett, B.S. (Columbia). Instr., Cooper Union, New York, N. Y.
V. C. Poor, Ph.D. (Chicago). Asst. prof., Univ. of Michigan, Ann Arbor, Mich.
B. H. Redditt, A.M. (Johns Hopkins). Asst., Johns Hopkins Univ., Baltimore, Md.
Mary Reichelderfer, M.S. (Chicago). Instr., St. Xavier Coll., Chicago, Ill.
J. F. Ritt, Ph.D. (Columbia). Asst. prof., Columbia Univ., New York, N. Y.
A. C. Robinson, A.B. (Pacific Univ.). Asst. prof., Ohio Univ., Athens, Ohio.
B. A. Rogers, M.S. (Chicago). Asst. prof. of physics, Conn. Agric. Coll., Storrs, Conn.
L. J. Rouse, Ph.D. (Michigan). Asst. prof., Univ. of Michigan, Ann Arbor, Mich.
Beulah Russell, A.M. (Chicago). Adj. prof., Randolph-Macon Woman's Coll., Lynchburg, Va.
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E. R. Smith, A.M. (Vanderbilt). Prof., Math. and physics, Fla. State Coll. for Women, Tallahassee, Fla.
V. A. Suydam, Ph.D. (Princeton). Prof., Physics and appl. math., Grinnell Coll., Grinnell, Iowa.
Anna H. Tappan, Ph.D. (Cornell). Asso. prof., Ia. State Coll., Ames, Ia.
T. Y. Thomas, Ph.D. (Princeton). Natl. Res. Fellow, Math. physics, Univ. of Chicago, 1923–1924.
Ruth Thompson, B.S. (N. J. Coll.). Instr., N. J. Coll. for Women, New Brunswick, N. J.
M. H. Trytten, A.B. (Luther). Physics, Luther Coll., Decorah, Ia.
J. S. Turner, Ph.D. (Chicago). Asst. prof., Ia. State Coll., Ames, Ia.
M. H. Tyler, B.S. (Amherst). Prof., R. I. State Coll., Kingston, R. I.
Helen M. Walker, A.M. (Columbia). Instr., Univ. of Kansas, Lawrence, Kans.
W. W. Wallace, A.M. (Stanford). Instr., Jr. Coll., Sacramento, Calif.
L. E. Ward, A.M. (Harvard). Instr., Rice Inst., Houston, Tex.

J. M. WEST, A.M. (Hamilton). Asst. prof., State College, Pa.
A. MARIE WHELAN, Ph.D. (Johns Hopkins). Prof., Olivet Coll., Olivet, Mich.
IDA WHITE, A.B. (Idaho). Teacher, High school, Caldwell, Ida.
MADELINE A. WHITNEY, A.B. (Oklahoma). High sch., Enid, Okla.
JOSEPH WILCZEWSKI, A.M. (St. Louis). Prof., St. Xavier Coll., Cincinnati, Ohio.
F. G. WILLIAMS, A.B. (Middlebury). Instr., Penn. State Coll., State College, Pa.
SARA M. WOLFE. Student, School of educ., Univ. of Pa., Philadelphia, Pa.
ROSCOE WOODS, Ph.D. (Illinois). Asso., State Univ. of Iowa, Iowa City, Ia.
ORPHA E. WORDEN, A.M. (Teachers' Coll., Columbia). Dept. of math., Detroit Teachers' Coll., Detroit, Mich.
ISABEL WORK, A.B. (Oklahoma). Asst. instr., Southeastern State Teachers Coll., Durant, Okla.
G. A. YANOSIK, C.E. (New York Univ.). Instr., New York Univ., New York, N. Y.
J. F. YOTHERS, A.M. (Otterbein). Head of dept. of math., Coe Coll., Cedar Rapids, Ia.

To institutional membership.

THE UNIVERSITY OF OKLAHOMA, Professor S. W. Reaves, Official representative.

W. D. CAIRNS, *Secretary-Treasurer.*

THE MAY MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The thirteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Association was held at Johns Hopkins University, Baltimore, Maryland, on Saturday, May 12, 1923. The Chairman of the Section, Professor FRANK MORLEY, presided at both morning and afternoon sessions.

There were forty-four in attendance, including the following twenty-eight members of the Association: O. S. Adams, H. G. Avers, Clara L. Bacon, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, Harry English, H. W. Ficken, W. M. Hamilton, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, L. T. Moore, Frank Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, R. E. Root, W. F. Shenton, C. A. Shook, Miss A. M. Whelan, E. W. Woolard. Those in attendance at the meeting were guests of the Johns Hopkins University for lunch.

The officers elected for the year 1923–1924 are: G. R. CLEMENTS, Chairman; HARRY ENGLISH, Secretary-Treasurer; FLORENCE P. LEWIS and J. J. LUCK, additional members of the executive committee. The fourteenth regular meeting of the Section will be held at Annapolis, Maryland, on Saturday, December 8, 1923.

The following papers were presented:

(1) "An application of tensor analysis to dynamics" by Professor F. D. MURNAGHAN;

(2) "Periodic functions in machine design" by Professor R. E. ROOT;

(3) "Invariants of the binary octavic" by Miss A. M. WHELAN (introduced by Professor Morley);

(4) "The volume bounded by Steiner's quartic surface" by Professor J. A. BULLARD;

(5) "The lemniscate functions as developed by Gauss" by Mr. O. S. ADAMS;

(6) "The motion of a particle on a rotating sphere; MONTHLY, Problem No. 2956 (*1922*, 81)" by Mr. W. D. LAMBERT;

(7) "An example of conformal mapping" by Professor FRANK MORLEY.

Abstracts of six of these papers follow:

1. Professor Murnaghan discussed the tensor form of the canonical equations of classical dynamics. Adopting the usual notation, the state of a dynamical system is indicated by a representative point in a *state-space* of $2n + 1$ dimensions (n being the number of degrees of freedom of the system) whose coördinates are (q, p, t). This space must not be confused with the more usual phase-space of Gibbs which separates out the time variable t from the others. Then the history of the dynamical system is represented by a curve in the state-space, which is an extremal for the line integral

$$\int p_1 dq_1 + \cdots + p_n dq_n - H dt,$$

where H is the Hamiltonian Function. The dynamical equations are merely the first Pfaff's system for the linear differential form; hence they state the vanishing of a covariant tensor of rank one. Classical dynamics conflicts with Relativistic geometry at the point where the latter demands that all physical equations be tensor equations in the *physical space-time continuum of four dimensions*, and not merely in the idealistic representative space.

2. Professor Root considered functions arising in problems in machine design that are expressed as algebraic functions of the sine or cosine of an angle. It is profitable to express such functions as Fourier series for convenience in computing their values, and to reveal their character in relation to vibrations. The representation in Fourier series is easily found, in most cases, by expanding the function as a power series in the sine or cosine, then expressing the successive powers in terms of functions of multiple angles and collecting coefficients. Professor Root contrasted this method of expanding in Fourier series with those usually treated in mathematical textbooks.

4. Professor Bullard called attention to the problem found in Granville's "*Elements of Differential and Integral Calculus*, rev. ed.," p. 420: Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. The speaker proposed this problem to the Monthly (*1920*, 326) and an incorrect answer was published (*1922*, 39).

If positive signs are taken before the square roots, no volume is enclosed by the portion of the surface where x, y and z are all less than a; but the rationalized equation is the equation of a Steiner's quartic surface.

When a regular tetrahedron of reference is chosen the Steiner's quartic, as a

model shows, is symmetrical, consists of four lobes of finite extent, and the three double lines intersect at right angles.

For the above equation the three coördinate planes and the plane at infinity form the tetrahedron of reference. Thus the surface lies entirely in the first octant (except that the double lines extend into the octants where just two coordinates are negative), three lobes are infinite in extent and one only encloses a finite volume. The three double lines intersect at (a, a, a) and are respectively perpendicular to the coördinate axes. The required volume lies entirely within a cube of edge a with opposite vertices at $(0, 0, 0)$ and (a, a, a) and may be considered as enclosed by $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$, $x^{1/2} + y^{1/2} - z^{1/2} = a^{1/2}$, $x^{1/2} - y^{1/2} + z^{1/2} = a^{1/2}$, $-x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. Computing the space in the corners and subtracting from a^3, we have for the required volume

$$a^3 - (a^3/90) - 3 \times (19/90)a^3 = (16/45)a^3.$$

Similarly for $(x/a)^{1/2} + (y/b)^{1/2} + (z/c)^{1/2} = 1$ we have $(16/45)abc$.

5. Mr. Adams pointed out that Gauss took the integral $\phi = \int_0^x (dx/\sqrt{1 - x^4})$ and inverted it by setting $x = \sin \text{lemn } \phi$, called the lemniscate sine of ϕ. Gauss noted in his diary "Functiones lemniscaticas considerare coeperamus, 1797, Jan. 8." This is the first example of the inversion of an integral to obtain a single valued function by means of which to study the properties of the many valued function defined by the integral. The paper identified the auxiliary functions of Gauss with the sigma functions of Weierstrass, limited to the special case of the lemniscate functions. It is to be regretted that Gauss never prepared his work for publication. As it is, we have no more than the list of various results, mainly without proof, given in the third volume of his collected works.

6. Problem 2956 (*1922*, 81) was discussed by the proposer, Mr. Lambert. The expression for the poleward force there given, $m\omega^2 \sin \phi \cos \phi$, should be corrected by multiplying it by the radius of the sphere. The problem is one that has long interested meteorologists as tending to throw light on the much more complex problem of atmospheric circulation. The compound centrifugal acceleration, which causes a moving particle to be deflected toward the right in the northern hemisphere, has long been known and its effects studied. For a small range of latitude the path of a particle moving under the action of no forces is approximately a small circle, as may be seen from elementary considerations. The rigorous expressions for the path of the particle involve elliptic functions. The path is not exactly a circle but a series of loops which are nearly coincident with one another for moderate velocities of projection, except near the equator. For high velocities and for regions near the equator the loops separate considerably and the possible paths resemble the various forms of the elastica.

7. In Professor Morley's note the mapping of a quadrangle on a rectangle was considered. It was shown that when the quadrangle is a parallelogram, if two opposite sides be levels and the other two sides stream lines, the flex-loci for all levels and all stream lines are the maps of the lines of symmetry of the rectangle.

G. R. CLEMENTS, *Secretary-Treasurer.*

THE APRIL MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section was held at Carleton College, Northfield, on Saturday, May 19, 1923. There were morning and afternoon sessions and at one o'clock luncheon was served in the men's dormitory of Carleton College. In the absence from the state of the regular chairman of the Section, Prof. C. H. Gingrich of Carleton College presided at both sessions.

The attendance was fifty-one including the following sixteen members of the Association: R. W. Brink, W. H. Bussey, H. W. Chandler, H. H. Dalaker, Gladys Gibbens, C. H. Gingrich, W. L. Hart, D. Jackson, R. A. Johnson, W. H. Kirchner, P. H. Nygaard, A. Reuterdahl, R. R. Shumway, Ella Thorp, H. B. Wilcox, Marion B. White.

The following officers were elected for the coming year: Chairman, H. H. DALAKER, College of Engineering, University of Minnesota; Secretary, R. W. BRINK, College of Science, Literature and the Arts, University of Minnesota; Executive Committee, C. H. GINGRICH, Carleton College, C. A. RUPP, Hamline University, GLADYS GIBBENS, University of Minnesota. A motion was adopted expressing the appreciation of the section for the cordial hospitality of Carleton College in entertaining the section at this meeting.

The following eight papers were read:

(1) "Triangular charts" by Professor W. H. KIRCHNER, University of Minnesota.

(2) "Some theorems in paper folding" by Mr. C. A. RUPP, Hamline University.

(3) "Galois fields" by Professor W. H. BUSSEY, University of Minnesota.

(4) "The significance of mathematical fundamentals" by President ARVID REUTERDAHL, Ramsey Institute of Technology.

(5) "The method of moments" by Professor DUNHAM JACKSON, University of Minnesota.

(6) "Dimensional analysis" by Professor J. T. TATE, University of Minnesota.

(7) "Eclipse problems and the total solar eclipse of Sept. 10, 1923," by Professor FATH, Carleton College.

(8) "Circles of antisimilitude of the circles through four points" by Professor R. A. JOHNSON, Hamline University.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. In his paper Prof. Kirchner showed how a certain proposition of elementary geometry is used as the basis of a system of nomographic charts, and how the construction of the charts affords much material for excellent examples of exercises in the teaching of mathematics.

Triangular charts with linear, functional and logarithmic scales were discussed. The gradual elaboration of the chart from a simple summation chart to a multiplication table of the form of $f_1(x^p)f_2(y^q)f_3(z^r) = C$ was illustrated. Further elaboration was indicated by the superposing of additional pencils on the original network. Devices for extending the scope and range of the charts, by the use

of movable graduated scales, or movable figures on millimeter and tracing papers, were discussed.

2. The method of generating a parabola by folding a piece of paper suggested to Mr. Rupp a possible generalization, which turned out to be a birational point-line transformation applicable to any curve. The inverse transformation has been called a *Podoid* by Brocard. Five theorems were stated, dealing with the mechanical construction by paper-folding of homothetic transformations, pedals and negative pedals, podoids and negative podoids, and a simple way of constructing an ellipse or hyperbola. Developments of these general ideas were suggested as possible for spaces other than flat, as non-Euclidean or spherical, or simple three-space.

3. Professor Bussey's paper was an elementary exposition of Galois fields designed especially for those not familiar with any finite fields other than the modular fields of prime order.

4. President Reuterdahl's paper contained a warning against too ready an acceptance of certain parts of mathematical theory as a correct interpretation of the physical universe.

5. Professor Jackson's paper was published in the MONTHLY for September-October, pp. 307–311.

6. Professor Tate applied the fundamental theorem of Dimensional Analysis to certain examples in physics, and showed how it may be used as a check and guide to the investigator.

7. Professor Fath showed slides of the various types of instruments used in eclipse observations and of the solar corona as shown at times of spot maximum and spot minimum.

8. Professor Johnson's paper appeared in full in the MONTHLY for July-August, pp. 250–252.

R. W. BRINK, *Secretary.*

THE DEVELOPMENT OF "PARTITIO NUMERORUM," WITH PARTICULAR REFERENCE TO THE WORK OF MESSRS. HARDY, LITTLEWOOD AND RAMANUJAN.[1]

By AUBREY J. KEMPNER, University of Illinois.

Part I.

HISTORICAL SKETCH LEADING UP TO THE WORK OF HARDY, LITTLEWOOD AND RAMANUJAN.

Introduction. In the last few years the genius of a small group of mathematicians has enriched the additive theory of numbers by new and powerful methods. At present we merely mention the names of these authors: G. H. Hardy (English) and J. E. Littlewood (English) have developed the bulk of the theory; the first papers were under the joint authorship of Hardy and S. Ramanujan (Hindu, *died*, 1920). A very important step in the theory could only be taken by making use of an independent investigation by H. Weyl (German).

[1] Among the references to the literature of the subject, the following are to be noted:

(*a*) *Sources of reference for "partitio numerorum" in general and for the main divisions of the subject,*

The papers in which the new theory is exposed are, in the main, of highly technical character. It is hoped that the present report, partly historical and partly synoptic, may indicate to readers, who are not primarily interested in the theory of numbers, the character of the new methods. A certain knowledge of the historical background is necessary for any proper appreciation of the new work; and since the problems with which the additive theory of numbers is concerned are not widely known, it seems justifiable to give a condensed sketch of the development of the subject.

1. In a rather vague manner, a distinction is often made in the theory of numbers between the "multiplicative" theory and the "additive" theory. For example, the theorem $a^p - a \equiv 0 \bmod p$ for any integer a and any prime p

(1) Bachman, P., *Niedere Zahlentheorie*, Zweiter Teil: *Additive Zahlentheorie* (1910).

(2) Dickson, L. E., *History of the Theory of Numbers*, vol. II (1920). Ch. III: Partitions; Chs. VI, VII, VIII; Ch. IX: Sums of Squares; Ch. XXV: Waring's problem.

(3) *Encyklopädie der mathematischen Wissenschaften*, I C 3, Bachmann, P., *Analytische Zahlentheorie.*

(4) *Encyclopédie des Sciences mathématiques*, I 17, Bachmann-Hadamard-Maillet, *Propositions transcendantes de la théorie des nombres.* Netto-Vogt, I 2, *Analyse combinatoire et théorie des déterminants.*

(5) MacMahon, P. A., *Combinatory Analysis*, vol. I (1915), II (1916).

(6) Netto, E., *Lehrbuch der Combinatorik* (1901).

(7) Sylvester, J. J., *Mathematical Papers*, vol. II (1908), 120–175. "Outlines of seven lectures on the partition of numbers." (Delivered 1859.)

(*b*) *Papers by* G. H. Hardy, J. E. Littlewood *and* S. Ramanujan *on the new method of dealing with problems of "partitio numerorum."* This list has been completed by taking into account the literature references contained in Siegel's (**28**) list.

(8) Hardy, "Asymptotic formulæ in combinatory analysis." *Comptes rendus du quatrième congrès des mathématiciens Scandinaviens à Stockholm* (1916), pp. 45–53.

(9) Hardy and Ramanujan, "Une formule asymptotique pour le nombre des partitions de *n*." *Comptes rendus de l'Académie des Sciences*, Paris, vol. 164 (1917), pp. 35–38.

(10) Ramanujan, "On certain trigonometric sums and their application in the theory of numbers." *Transactions of the Cambridge Philosophical Society*, vol. 22 (1918), pp. 259–276.

(11) Hardy, "On the coefficients in the expansion of certain modular functions." *Proceedings of the Royal Society*, London, A 95, 1918, pp. 144–155.

(12) Hardy and Ramanujan, "Asymptotic formulæ for the distribution of integers of various types." (Read May, 1916.) *Proceedings of the London Mathematical Society*, ser. 2, vol. 16 (1918), pp. 112–132.

(13) Hardy and Ramanujan, "Asymptotic formulæ in combinatory analysis." (Read Jan. 1917.) *Proceedings of the London Mathematical Society*, ser. 2, vol. 17 (1918), pp. 75–115.

(14) Hardy, "On the expression of a number as the sum of any number of squares, and in particular of five and seven." *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 189–193.

(15) Hardy and Littlewood, "A new solution of Waring's problem." *Quarterly Journal of Mathematics*, vol. 48 (1919), pp. 272–293.

(16) Hardy and Littlewood, Note on Messrs. Shah and Wilson's paper entitled "On an empirical formula connected with Goldbach's Theorem." *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 245–254.

(17) Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five." *Transactions of the American Mathematical Society*, vol. 21 (1920), pp. 255–284.

(18) Hardy, "Some famous problems of the theory of numbers and in particular Waring's problem." An inaugural lecture delivered before the University of Oxford. Oxford, 1920.

represents a "multiplicative" property; the theorem that every positive integer is either a cube, or the sum of two positive cubes, or the sum of three, or four, . . ., or nine positive cubes, will be considered as belonging to the "additive" theory.

Since the time of Legendre and Gauss—particularly since 1801, the year in which appeared Gauss' monumental *Disquisitiones Arithmeticæ*—the systematic development of the theory of numbers has been predominantly along the lines of the multiplicative theory. It must however not be thought that no important results are known in the additive theory. Indeed, there is a profusion of dazzling isolated theorems and groups of theorems.[1] If we merely count interesting theorems, we find that the additive theory is perhaps not much weaker than

A series of papers, under the collective title: Hardy and Littlewood, Some problems of "Partitio Numerorum."

(19) I. "A new solution of Waring's problem." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1920, pp. 33–54.

(20) II. "Proof that every large number is the sum of at most 21 biquadrates." *Mathematische Zeitschrift*, vol. 9 (1921), pp. 14–27.

(21) III. "On the expression of a number as a sum of primes." *Acta Mathematica*, vol. 44 (1922) pp. 1–70.

(22a) IV. "The singular series in Waring's problem and the value of the number $G(k)$." *Mathematische Zeitschrift*, vol. XII, 1922, pp. 161–188. (Reprinted from *Hilbert Festschrift*, Jan. 1922, pp. 365–392.)

(22b) V. "A further contribution to the study of Goldbach's problem." *Proceedings of the London Mathematical Society*, ser. 2, vol. 22 (1923), pp. 46–56.

(c) *Papers on the new method by other mathematicians*,

(23a) *Fortschritte der Mathematik*, vol. 46 (covering the years 1916–18), pp. 200, 201.

(23b) Carmichael, R. D., *Bulletin of the American Mathematical Society*, vol. 27 (1921), 471–5 (Review of (18)).

(24) Landau, E., "Über die Hardy-Littlewoodschen Arbeiten zur Additiven Zahlentheorie." *Jahresbericht der Deutschen Mathematiker Vereinigung*, vol. 30 (1921), pp. 179–185.

(25) Landau, E., "Zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1921, pp. 88–92.

(26) Landau, E., "Zum Waringschen Problem." *Mathematische Zeitschrift*, vol. XII (1922), pp. 219–247. (Reprinted from *Hilbert Festschrift*, Jan. 1922, pp. 422–451.)

(27) Ostrowski, A., "Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Mathematische Zeitschrift*, vol. IX (1921), pp. 28–34.

(28) Siegel, C. L., "Additive Theorie der Zahlkörper I." *Mathematische Annalen*, vol. 87 (1922), pp. 1–35.

(29) Weyl, H., "Über die Gleichverteilung von Zahlen mod. 1." *Mathematische Annalen*, vol. 77 (1916), pp. 313–352.

(30) Weyl, H., "Bemerkung über die Hardy-Littlewoodschen Untersuchungen zum Waringschen Problem." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1921, pp. 189–192.

(31) Weyl, H., "Bemerkung zur Hardy-Littlewoodschen Lösung des Waringschen Problems." *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen, 1922.

Numbers (8), (21), (22 b), (25), (30), (31) were not available to the writer.

Reference of the text to the preceding literature is made by corresponding numbers. Other text references are given by footnotes.

As preparation for a detailed study of the new theory, the (partly) synoptic papers (15), (18), (23), (24) should be of great value.

In (26) Landau derives *ab ovo* the main results of Hardy and Littlewood for Waring's problem, with original contributions and modifications.

It is strongly to be hoped that American mathematicians will contribute their share toward the development of the new field.

[1] (1), (2).

the multiplicative. But while in the multiplicative theory the theorems all find their proper place in a beautiful system, as do the building stones in an imposing structure, the theorems of the additive theory may be compared to a vast accumulation of building material for a palace which is at present barely planned.

It is also true that a large number of theorems, which by their enunciation should be counted to the additive theory, at present derive their proof from methods belonging to the multiplicative theory.

2. One of the most important branches of the additive theory, and one which contains in itself an inexhaustible supply of large problems and groups of problems, is the "partitio numerorum," or the theory of the partition of positive integers into positive summands. This name is probably due to L. Euler, who called chapter XVI of his famous *Introductio in Analysin Infinitorum*,[1] "De Partitione Numerorum."

As starting point we may consider the problem of determining the total number $p(n)$ of ways in which a given positive integer n may be broken up into positive integral summands, counting as identical two partitions which are distinguished only by the order of the summands; thus $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$, $p(5) = 7$.

This problem had already been suggested about a century earlier by Leibniz, to one of the Bernoullis.

The range of problems belonging to partitio numerorum is much wider than might be at first thought. Instead of admitting as summands the range of all positive integers, we may select any particular, finite or infinite, range of positive integers. As of interest for our purposes, we mention the following ranges:

1. All positive integers, 1, 2, 3, $\cdots$,
2. A finite set of integers, for example, 1, 2, 3,
3. All odd numbers, 1, 3, 5, $\cdots$,
4. All odd primes and the number 1,
5. The "triangular" numbers, 1, 3, 6, 10, $\cdots$,
6. The "square" numbers, 1, 4, 9, 16, $\cdots$,
7. The "k-gon" numbers for a given integer $k \geqq 3$ (the nth k-gon number $= \frac{1}{2}n(n-1)(k-2) + n$),
8. The cubes 1, 8, 27, $\cdots$,
9. The kth powers, for a given positive integer k, 1^k, 2^k, 3^k, $\cdots$.

The original problem (1 above) would then be modified into the determination of the number of ways in which (for example, for 6. above) the positive integer n may be written as a sum of positive squares (as: $10 = 1^2 + 1^2 + \cdots + 1^2 = 2^2 + 1^2 + \cdots + 1^2 = 2^2 + 2^2 + 1^2 + 1^2 = 3^2 + 1^2$, $p(10) = 4$).

[1] *Introductio in Analysin Infinitorum*, vol. I, 1748. It may be noted that Ch. XV of Euler's work, which deals with certain types of infinite series, has also been of great influence in the development of the theory of numbers. Many results which were stated by Euler without proof have yielded to treatment only in our generation. As a sample we mention that a certain series whose terms depend on the prime numbers and which starts with $1 + 1/3 - 1/5 + 1/7 + 1/9 + 1/11 - 1/13 - 1/15 \cdots$ converges to the value $\pi/2$. See E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1909, vol. II, pp. 645 ff.

3. A very important group of problems arises when we restrict the number of summands to be used in the representation of the number n; for example, when we ask for the number of ways in which n may be the sum of exactly five positive squares, or the sum of exactly nine positive cubes, etc. Equally important is the question concerning the number of ways in which n can be represented as the sum of *not more* than a given number k of elements of the set; for example, the number of ways in which n may be the sum of *five or fewer* positive squares, of *nine or fewer* positive cubes, etc. It may further be specified that the same summand may be used only once, or may be repeated. Thus, in how many ways can n be the sum of (exactly) five *distinct* positive squares; of five or fewer *distinct* squares? These remarks will serve to indicate the bewildering complexity of the group of problems under consideration.

A very general type of problem then is the following: *Given a set of positive integers* $c_1 < c_2 < c_3 < \cdots$, *finite or infinite in number, to find, for a given positive integer n, the number of ways in which n may be formed of summands chosen from the c's, be it that the number of summands is limited, or is not limited, be it that the summands may be repeated or must be distinct.*

Every such problem may be expressed in the form of a Diophantine equation: for example, the number of positive solutions of $n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ gives the number of ways of expressing n as a sum of five positive squares. In $n = 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3$ the number of (non-negative) solutions gives the number of ways in which n may be built up from 1, 2, 3, allowing repetitions. Thus, $p(7) = 8$, corresponding to the eight solutions of $1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 = 7$: (7, 0, 0), (5, 1, 0), (4, 0, 1), (3, 2, 0), (2, 1, 1), (1, 3, 0), (1, 0, 2), (0, 2, 1). However, we do not possess any powerful general method of attacking the problems from this direction.

4. We remember that we did not count as distinct two partitions which differ only in the order of their summands; for example, $1 + 2 + 1 + 2$ and $1 + 1 + 2 + 2$, would not be counted separately. When the order of the summands is to be taken into account, one sometimes speaks of "compositions" instead of "partitions," so that the problem of determining the number $c(n)$ of compositions of n means to find the number of partitions, but to count each partition with a multiplicity measured by the number of ways in which the order of the summands may be chosen. For $n = 3$, $c(n) = 4$, from $3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$.

The problems of composition would seem to be in some respects of less importance than the problems of partition. It does not usually seem to be possible to derive, from the solution of a problem of composition, the solution of the corresponding problem of partition. Quite an elaborate theory of the composition of integers into sums of squares is in existence.

We shall restrict our attention essentially to the problems of partition. Problems of partition and composition can also be expressed in a natural fashion as problems of the determination of the number of "lattice-points" in certain regions—"Gitterpunkt Anzahlen." To mention only two examples: The num-

ber of *compositions* of n into two squares is obviously equal to the number of lattice-points (points whose coördinates are integers or zero) lying on the circumference of a quadrant of the circle $x^2 + y^2 = n$ in the (xy)-plane. The number of *compositions* of n into three or fewer squares is represented by the number of lattice-points lying on the surface of the sphere $x^2 + y^2 + z^2 = n$ within the first octant of the (xyz)-space; while the number of *partitions* of n into three squares would be given by the number of lattice-points lying on a certain smaller portion [1] of the surface of the sphere. Similarly for a composition involving a set of k summands (for example, the representation of a number as the sum of nine non-negative cubes) an illustration in k-dimensional space (nine dimensional, working with the "surface" $x_1^3 + x_2^3 + \cdots + x_9^3 = n$) can be given.

However, the problem of finding such "Gitterpunkt Anzahlen" to the degree of accuracy necessary in order to be useful in partitio numerorum is, in practically all cases, one of desperate difficulty.

5. If $p(n) = 0$, in a given problem of partitions, there is no representation of n in the required form while, when $p(n) > 0$, there is at least one representation. The further questions thereby arise:

Which numbers [2] *n can be and which cannot be represented in the desired form? How must an infinite set of numbers be chosen in order that every n can be represented as a sum of a fixed number of elements of the set?*

We shall assume, throughout, that every element may be used repeatedly, as in $7 = 2^2 + 1^2 + 1^2 + 1^2$.

Concerning the first question, a large number of more or less isolated theorems are known. To mention some classical results concerning the representation of integers as sums of powers:

1. A number is the sum of two positive squares when, and only when, it contains no prime factor of the form $4k + 3$ to an odd power.

2. A number is the sum of three or fewer positive squares when, and only when, it is not of the form $4^\alpha \cdot (8k + 7)$, $\alpha \geqq 0$.

3. Every number is the sum of four or fewer positive squares.

4. Every number is the sum of nine or fewer positive cubes.

Concerning the second question no very deep-lying results seem to be known, in the sense of establishing necessary and sufficient conditions for a set of integers to have the property indicated.[3]

There are important reasons why one should in many cases restrict one's self in problems of partition to the representation of large numbers n: this tends to eliminate what may be called disturbing individual properties of the separate c's. The most clearly outstanding results of the new work of Hardy, Littlewood and Ramanujan are in this direction.

[1] Of area 1/48 of the total surface of the sphere.

[2] We frequently write "number" for "positive integer," where no misunderstanding is to be feared.

[3] The following *necessary* condition is not deep-lying and is easily established: *If* $0 < c_0 < c_1 < c_2 < \cdots$ *are an infinite set of integers possessing the property that every positive integer can be represented as the sum of a limited number of the set, then the power series* $c_0 + c_1 x + c_2 x^2 + \cdots$ *has the unit circle for its circle of convergence.*

It is essential, in problems of partition, at least at present, that we consider only addition of the positive elements c_r, not subtraction. Therefore Goldbach's conjectured theorem: *Every even number is a sum of two primes*, is properly a problem of partition, but Fermat's conjectured theorem: $x^n + y^n = z^n$ *is not solvable in positive integers* x, y, z *for* $n > 2$ does not belong to this domain, because the problem would require discussion of the equation $0 = -z^n + x^n + y^n$.

It should be remarked that our most powerful methods of attack on Fermat's problem are of distinctly multiplicative character.

6. Already Euler (*loc. cit.*) connected the most important types of problems of partition with certain infinite series and infinite products, the so-called generating functions of the problem, and since most of the serious attacks on the problems of partition which have been made since Euler's time have used this method of generating functions as a starting point, and since, in particular, the new methods of Hardy, Littlewood, and Ramanujan consist in an extreme sharpening of this weapon, we must indicate in some examples this method.

I. *To find the number* $N(n)$ *of partitions, without repetition, of a number* n: $(7 = 6 + 1 = 5 + 2 = 4 + 3 = 4 + 2 + 1, N(7) = 5)$.

The problem might also be formulated: *To find the number of solutions of* $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n$, $x_i = 0, 1$. But, as already indicated, we possess no means of treating this Diophantine equation directly.

Euler attacks the question as follows without, it may be stated, paying any attention to questions of convergence, etc.:

Let

$$\prod_{i=1}^{\infty}(1 + x^i) = (1 + x^1)(1 + x^2)(1 + x^3)\cdots = 1 + c_1x^1 + c_2x^2 + \cdots + c_nx^n + \cdots.$$

The number of partitions required is clearly given by c_n (for example, $c_7x^7 = x^7 + x^{6+1} + x^{5+2} + \cdots + x^{4+2+1} = 5x^7$, $c_7 = 5$).

It goes without saying that Euler did not possess modern function-theoretic methods for determining the value of c_n, nor did he possess any other powerful methods of dealing with this and other partition problems. To us is of great interest the fact that he was apparently the inventor of this method of "generating functions."

II. *To find the number of partitions of* n, *with repetition.* For example, $p_5 = 7$. The corresponding Diophantine equation would be $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = n$, $x_i \geqq 0$. This is again useless. Euler starts out with

$$\begin{aligned}\prod_{i=1}^{\infty}\frac{1}{1 - x^i} &= (1 + x^1 + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 \\ &\qquad + x^6 + x^9 + \cdots)\cdots \textit{in inf.} \\ &= 1 + x^{1\cdot 1} + (x^{2\cdot 1} + x^{1\cdot 2}) + (x^{3\cdot 1} + x^{1\cdot 2+1\cdot 1} + x^{1\cdot 3}) + \cdots \\ &= c_0 + c_1x^1 + c_2x^2 + c_3x^3 + \cdots,\end{aligned}$$

and, again, the coefficient c_n of x^n gives the required number $p(n)$. Of course, this number grows tremendously large with increasing n. This problem we

shall discuss in some detail (sections **12–17**). At present we only state that one of Hardy's (and Ramanujan's) first results of the new theory was to obtain an accurate expression for $p(n)$, and one which can really be handled. In this problem, $\Pi(1-x^i)^{-1}$ is the generating function, and Hardy and Ramanujan teach how to deal with this very complicated infinite product, as an analytic function of (complex) x, in the neighborhood of the natural boundary of the function.

In order to show the difference between a problem of partition and the corresponding problem of composition, we treat the question:

III. *To find the number of compositions of n,*[1] *with repetition.* We determine first the number of compositions of n into exactly s positive summands. As generating function we may choose

$$\left(\frac{x}{1-x}\right)^s = (x+x^2+x^3+\cdots)^s = a_s x^s + a_{s+1}x^{s+1} + \cdots.$$

It is found that

$$\left(\frac{x}{1-x}\right)^s = x^s\left(1+\frac{s}{1}x+\frac{s(s+1)}{1\cdot 2}x^2+\cdots + \frac{s(s+1)\cdots(s+i-1)}{1\cdot 2\cdots i}x^i+\cdots\right) = \sum_{n=s}^{\infty}\binom{n-1}{s-1}x^n$$

and therefore $\binom{n-1}{s-1}$ is the number of compositions of n into exactly s positive summands. Hence the total number of compositions of n into positive summands is

$$\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-1} = 2^{n-1}.$$

For example, for $n = 3 : 3 = 2+1 = 1+2 = 1+1+1$. In order to pass from this problem to II, it would be necessary first to know how many partitions of n have a group of exactly n_1 summands alike, another group of n_2 summands alike, etc., when $n_1, n_2, \cdots$ are any positive integers such that $n_1 + n_2 + \cdots = n$. This would be a very difficult problem.

IV. *To find the integers which are the sum of exactly two odd primes p_i.* We may choose as generating function

$$(x^{p_1}+x^{p_2}+\cdots)^2 = c_1x^{\alpha_1}+c_2x^{\alpha_2}+\cdots,$$

where $p_1 < p_2 < \cdots$ are the successive odd primes. In particular, Goldbach's theorem (see section **5**) would be proved if we could show that for every even exponent the corresponding coefficient does not vanish. Of this theorem, E. Landau, author of the standard work on prime numbers, the "Handbuch" already referred to, said, in 1912, that he considered this problem "unangreifbar beim gegenwärtigen Stande der Wissenschaft."[2] Nothing perhaps can better

[1] **(1)**, **(2)**.

[2] International Congress, Cambridge, 1912: Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung.

emphasize the importance of the new work than the fact that this problem is, in a perfectly definite sense, brought under the range of Hardy and Littlewood's heavy artillery so that we may, without undue optimism, hope to see it solved in our generation.

Hardy and Littlewood actually prove[1] that every sufficiently large odd positive integer is the sum of at most three primes, provided that we admit a certain generalization of a famous assumption, which has never been either proved or disproved, concerning the zeros of the Riemann ζ-function, of which an expansion is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

V. For a special purpose, we also indicate Jacobi's proof (not historically the first one) of the theorem that every positive integer is the sum of four non-negative squares. As generating function one might choose $(1 + x^{1^2} + x^{2^2} + \cdots)^4$. In his *Fundamenta nova*,[2] Jacobi—who always kept his eye open for mathematical by-products—made use of the fact that he was able to express a certain quantity entering in his theta-series in two different manners:

$$\left(\frac{2K}{\pi}\right)^2 = (1 + 2q^1 + 2q^4 + 2q^9 + \cdots)^4$$
$$= 1 + 8\left\{\frac{q}{1-q} + \frac{q^2}{1+q^2} + \frac{q^3}{1-q^3} + \frac{q^4}{1+q^4} + \cdots\right\}.$$

Obviously, $(1 + 2q^1 + 2q^4 + \cdots)^4$ may be used as generating function instead of $(1 + q^1 + q^4 + \cdots)^4$. The argument is now as follows:

$$\begin{aligned}
&(1 + 2q^1 + 2q^4 + 2q^9 + \cdots)^4 \\
&\quad = 1 + 8(q + q^2 + q^3 + q^4 + q^5 + q^6 + + + \cdots) \\
&\qquad + 8(+ q^2 \quad - q^4 \quad + q^6 - + - \cdots) \\
&\qquad + 8(\quad + q^3 \quad + q^6 + + + \cdots) \\
&\qquad + 8(\quad + q^4 \quad - + - \cdots) \\
&\qquad + 8(\quad + q^5 \quad + + + \cdots) \\
&\qquad + 8(\quad + q^6 - + - \cdots) \\
&\qquad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad = 1 + 8(1\cdot q + 2\cdot q^2 + 2\cdot q^3 + 1\cdot q^4 + 2\cdot q^5 + 4\cdot q^6 + \cdots),
\end{aligned}$$

and it is immediately clear that, for α prime, q^α has the coefficient 2, that is, a value different from zero. Therefore, certainly every *prime* number is the sum of four or fewer positive squares, and using a classical elementary identity,[3] every positive integer is seen to possess this property. In this case we succeed because we know enough concerning the properties of the function represented

[1] (**21**), (**22b**). For summary of (**22b**) see footnote on first page of Part II.

[2] *Fundamenta nova theoriae functionum ellipticarum*, 1829, p. 188.

[3] $(a^2 + b^2 + c^2 + d^2)\cdot(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = A^2 + B^2 + C^2 + D^2,$

where

$$A = a\alpha - b\beta - c\gamma + d\delta, \quad B = a\beta + b\alpha - c\delta - d\gamma,$$
$$C = c\alpha + a\gamma + d\beta + b\delta, \quad D = d\alpha - a\delta - c\beta + b\gamma.$$

by $(1 + 2q^1 + 2q^4 + \cdots)^4$. Except for certain simple types of generating functions (rational functions) we must admit that this is something very exceptional. Some of Hardy's important papers on the new theory[1] are closely connected with certain related problems. He gives also references to the many important papers which have been written on the problem of breaking up a number into a given number of positive squares, based on the theory of theta-series.

VI. As a last example of generating functions we select Waring's problem. In 1770, Waring stated,[2] without any attempt at proof, an extension to higher powers of the theorem that every number is equal to the sum of four non-negative squares. Waring's statement is: *For every positive integral exponent k there exists a positive integer g_k, depending only on k, such that every positive integer is the sum of g_k or fewer positive kth powers.* He stated that for example every natural number is the sum of nine or fewer cubes; of nineteen or fewer fourth powers, and other cases. That at least nine cubes are required follows from the fact that $23 = 2 \cdot 2^3 + 7 \cdot 1^3$ cannot be represented by fewer cubes; similarly, $79 = 4 \cdot 2^4 + 15 \cdot 1^4$ requires nineteen fourth powers.

Starting about the middle of last century, the finiteness of g_k was gradually proved by means of elementary, but sometimes very complicated, modifications of the theorem $g_2 = 4$ and of similar theorems, first for $k = 4$, then for $k = 3, 5, 6, 7, 8, 10, 12, 14$,[3] and finite upper bounds for the g_k in these cases were obtained or indicated. The general existence proof of g_k for all positive integers k was established by Hilbert, in 1909, in a famous paper.[4] The main part of Hilbert's proof is of transcendental character, since it is based on the transformation of a certain quintuple integral (25-fold in the first presentation, in the *Nachrichten der Gesellschaft der Wissenschaften*, Göttingen). The transcendental character of Hilbert's proof has been gradually eliminated by various authors, but it still has the character of a pure existence proof. It is in connection with Waring's problem that Hardy and Littlewood have made the most fundamental and revolutionizing progress.[5] If we are to use generating functions in Waring's problem, the problem would, for the case $k = 3$, present itself as follows:

In $f(x) = (1 + x^{1^3} + x^{2^3} + \cdots)^s = 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$, *how large must the exponent s be chosen in order that all c_n shall be > 0?*

As a matter of fact, for reasons already mentioned, one is much more interested in the number s which is large enough to ensure that all c_n for sufficiently large n are positive, thus determining a number G_3 such that every *sufficiently large*

[1] (14), (17).

[2] Waring, *Meditationes Algebraicae*, 1770.

[3] (1), (2), (18) or Kempner, *Das Waringsche Problem*, Diss. Göttingen, 1912.

[4] Mathematische Annalen, vol. 67 (1909), p. 281. Hilbert's proof does not give a method of actually determining an upper bound for g_k, however large, for any single k. (It is true that Hilbert gives some indications as to how an upper bound might be determined from his paper, but these suggestions have never been carried out. The results which might be thus obtained would certainly be incomparably less precise than the results obtained by Hardy and Littlewood in their new work.)

[5] (15), (18), (19), (20), (22), also (23), (24), (25), (26), (27), (28), (30), (31).

number may be represented as the sum of not more than G_3 cubes (and similarly for kth powers). From its definition, $G_k \leqq g_k$. This inequality is very interesting. There can be, from divers considerations, little doubt that for all $k \geqq 3$ really $G_k < g_k$, but the only cases for which this question is settled, even using all the new results of Hardy and Littlewood, is $G_2 = g_2 = 4$; $G_3 \leqq 8$, $g_3 = 9$.[1] If we were in possession of a theory of the series $(1 + x^{1^3} + x^{2^3} + \cdots)^s$ in the sense in which we happen to possess a theory of the series $(1 + 2x^{1^2} + 2x^{2^2} + \cdots)^4$ in the theory of elliptic functions, we might hope to discover some other expansion of the function from which we could learn something about our coefficients in $\sum c_n x^n$. However, we have no such theory; the elliptic function theory employs only functions of the type mentioned last. Hardy's and Littlewood's main contributions[2] to Waring's problem may be briefly said to consist in the following:

(1) They invent means of proving that, for $s > (k-2)2^{k-1} + 5$, at most a finite number of coefficients of $(1 + x^{1^k} + x^{2^k} + \cdots)^s = \sum c_n x^n$ vanish, that is, they prove that all sufficiently large numbers are a sum of $(k-2)2^{k-1} + 5$ or fewer positive kth powers.

(2) Beyond this they determine the "order of magnitude" of c_n as a function of n, thus determining for any given exponent k the *order of magnitude* of the number of different manners in which a given large number n may be written as the sum of s or fewer kth powers. It is reasonably safe to say that nobody had dreamed that our mathematical power was great enough to even attempt a problem like this.

7. We naturally ask whether no work was done, with the exception of the large but very special body of results connected with the theta-series, on the basis of Euler's generating functions during the century and a half between Euler's *Introductio* and Hardy's and Littlewood's and Ramanujan's accomplishments. The answer is that an enormous amount of work was done in this direction, first of a formal "Combinatory Analysis" character, incorporated by MacMahon into (5), then about the middle of last century powerful algebraic methods were applied by, among others, Cayley and Sylvester.[3] Also, in simple cases, the Cauchy calculus of residues was employed to determine the coefficient c_n of the expanded power series. At present, we illustrate these algebraic methods by an example which has been often used for this purpose, so by Hardy.[4] We do this to bring out two points.

[1] Since a cube is of one of the three forms $9k + 1$, $9k - 1$, $9k$, it is clear that every number of the form $9k \pm 4$ requires at least four positive cubes, *i.e.*, $G_3 \geqq 4$. A situation similar to the one revealed by the inequality $G_3 < g_3$ has been known to exist, since a century, in another problem. We consider the polygonal numbers of order m: $c_\mu = \frac{1}{2}\cdot\mu(\mu-1)(m-2) + \mu$; $c_0 = 0$, $c_1 = 1$, $c_2 = m$, $\cdots$. It is clear that the number $m - 1$ requires $m - 1$ summands from the set c_μ. Yet, Legendre (*Théorie des nombres*, vol. 2, 3d ed., 1830, p. 350) proved that every even number $> 28m^3$ is a sum of *four or fewer* c_μ's, every odd number $> 28m^3$ a sum of *five or fewer* c_μ's of which at most four need be chosen > 1. The extra summand 1 which appears in the statement for odd numbers is, not unlikely, only due to the method of proof.

[2] Hardy and Littlewood really do more; we refrain from quoting the formulæ embodying their results.

[3] (1), (2), (3), (4), (5), (6), (7).

[4] (13), (18).

Consider the number $N(n)$ of partitions, with repetition, of n into summands 1, 2, 3. For example, $N(5) = 5$.

A generating function is clearly

$$f(x) = (1+x^1+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots) = [(1-x)(1-x^2)(1-x^3)]^{-1}.$$

Breaking up into partial fractions:

$$f(x) = \frac{1}{6(1-x)^3}+\frac{1}{4(1-x)^2}+\frac{17}{72(1-x)}+\frac{1}{8(1+x)}+\frac{1}{9(1-\omega x)} + \frac{1}{9(1-\omega^2 x)}, \qquad \omega^2+\omega+1=0,$$

$$f(x) = \tfrac{1}{12}(1\cdot 2+2\cdot 3x+3\cdot 4x^2+\cdots)+\tfrac{1}{4}(1+2x+3x^2+\cdots) + \tfrac{17}{72}(1+x+x^2+\cdots)+\tfrac{1}{8}(1-x+x^2-\cdots) + \tfrac{1}{9}(1+\omega x+\omega^2x^2+\cdots)+\tfrac{1}{9}(1+\omega^2x+\omega^4x^2+\cdots),$$

from which

$$c_n = \frac{17}{72}+\frac{1}{8}(-1)^n+\frac{\alpha}{9}+\frac{1}{4}(n+1)+\frac{1}{12}(n+1)(n+2),$$

where $\alpha = \omega+\omega^2, \omega^2+\omega, 1+1$, according as $n \equiv 1, 2, 0 \bmod 3$, resp.;

$$c_n = -\frac{7}{72}+\frac{1}{8}(-1)^n+\frac{2}{9}\cos\frac{2n\pi}{3}+\frac{1}{12}(n^2+6n+5+4) = -\frac{7}{72}+\frac{1}{8}(-1)^n+\frac{2}{9}\cos\frac{2n\pi}{3}+\frac{1}{12}(n+3)^2.$$

This is the formula for $N(n)$, and it therefore represents a positive integer. Hence, from

$$\left|-\frac{7}{72}+\frac{1}{8}(-1)^n+\frac{2}{9}\cos\frac{2n\pi}{3}\right| < \frac{1}{2},$$

$N(n) =$ *integer closest to* $\frac{1}{12}(n+3)^2$.

The number which we have found is Sylvester's "denumerant,"

$$N(n) = \frac{n}{1,\ 2,\ 3}.$$

Similarly, the denumerant

$$N(n) = \frac{n}{a_1,\ a_2,\ \cdots,\ a_m}$$

represents the number of partitions of n into summands chosen from $a_1, a_2, \cdots, a_m$. The rudiments of a calculus of denumerants exist, but the formulæ are for the most part recurrence formulæ which do not give a satisfactory answer to the general problem of determining the value of $N(n)$. Jacobi, who, as we have already seen, also worked in the field of partitions, used for $\frac{n}{1,\ 2,\ 3}$ the notation $N(n = 1\cdot x_1+2\cdot x_2+3\cdot x_3),\ x_i \geqq 0$.

The following quotation from Sylvester will show the importance he attached to the theory of partitions: "Partitions constitute the sphere in which analysis lives, moves and has its being; and no power of language can exaggerate or point out too forcibly the importance of this till recently almost neglected but vast, subtle and universally permeating element of algebraic thought and expression." (Quoted by Bachmann, (1) p. 104.)

8. The two points to which we wish to call attention are these:

I. $\frac{n}{1, 2, 3}$ involves in a natural and, indeed, probably unavoidable manner complex roots of unity.

II. It consists of parts of two different types, one part monotonically increasing with n, the other part of oscillatory character.

These latter form essentially the famous "waves" of Sylvester.[1] In a rather formal manner these denumerants have been carefully examined and we must, in justice, say that the problem of finding the total number $p(n)$ of partitions of a number into positive summands was, in a certain sense, solved by such methods in 1914 by Csorba.[2] The formula is of elementary character, containing, as it does, only finite summations and products. Little more than this can be said in its favor. It would probably be impossible to obtain from it any information concerning the order of magnitude of $p(n)$; hopeless to attempt to read from it when or whether $p(n)$ has a value $\neq 0$; impossible to carry through the computations for any but the very smallest values of n.

9. Hardy, in his beautiful inaugural lecture (18), illustrates the new method of procedure in problems of partition by a brief discussion of the problem treated in section 7. It is first mentioned that the following method must have suggested itself not to one or two, but to many mathematicians. It amounts to the determination of the residue of an analytic function. We consider

$$f(x) = \frac{1}{(1-x^1)(1-x^2)(1-x^3)} = 1 + c_1x^1 + c_2x^2 + \cdots + c_nx^n + \cdots$$

as an analytic function of the complex variable x. Its sole singularities are simple poles at certain points of the circumference of the unit circle, namely, at $1, -1, \omega, \omega^2$ $(\omega^2 + \omega + 1 = 0)$. Therefore, by Cauchy's integral, c_n, the number of partitions required, is given by

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x^{n+1}}\,dx,$$

where the integration is carried around a circle Γ of radius $r < 1$ about the origin. This integral can be evaluated in the following manner: we deform the path as in Figures 1, 2, 3 (see Part II). Then

$$\int_\Gamma = \int_{\Gamma_1} - \int_{C_1} - \int_{C_2} - \int_{C_3} - \int_{C_4},$$

[1] (1), (2).

[2] *Mathematische Annalen*, 1914, vol. 75, pp. 545–568.

where we may choose the circle Γ_1 as large as we like, and where we assume that all integrals are taken in a positive sense (thus accounting for the four minus signs in the last formula). But on account of the denominator x^{n+1} in the integrand we are sure that, for sufficiently large values of n, $|\int_{\Gamma_1}| < \epsilon$, $\epsilon > 0$ arbitrarily small, so that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(x)}{x^{n+1}}dx = -\frac{1}{2\pi i}\left\{\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right\}\frac{f(x)}{x^{n+1}}dx.$$

The problem thus reduces to an exercise in Cauchy's Calculus of Residues.

10. *Cannot this method be extended to all problems of partition?* For example, to the problem II of section 6 (total number of partitions of n, with the generating function $\Pi_{i=1}^{\infty}(1-x^i)^{-1}$), or to the problem VI (Waring's problem, with a generating function $(1+x^{1^k}+x^{2^k}+\cdots)^s$)? We do not have to seek very far for an answer to this question. In cases where our generating function has a finite number of singularities, of so mild a character that we may hope to integrate $f(x)/x^{n+1}$ around each one separately, and if $f(x)$ exists in the entire complex plane and behaves for large values of $|x|$ in such manner that we may expect to integrate $f(x)/x^{n+1}$ around a large circle about the origin (or around any large closed contour extending far enough in all directions), we may hope to succeed by this method.

Unfortunately, these prerequisites are satisfied only in the simplest cases, notably when $f(x)$ is a simple rational function. This represents but a very small subclass of the totality of partition problems, and these not the most interesting or important.

Consider the two problems last mentioned (II and VI of section 6). In problem II, $f(x) = [(1-x^1)(1-x^2)(1-x^3)\cdots]^{-1}$ has obviously a singular point at every point on the unit circle representing a root of unity, that is, the singular points are everywhere dense on the circumference so that the unit circle is a natural boundary for the function, across which the function can in no plausible way be extended, quite apart from the fact that it would not be possible to integrate around the individual singularities. Similarly, in problem VI, it is known from general theorems in the theory of analytic functions that $(1+x^{1^k}+x^{2^k}+\cdots)$ and therefore also $f(x) = (1+x^{1^k}+x^{2^k}+\cdots)^s$ represents, for $k > 1$, a function which has the unit circle for a natural boundary. It is therefore to be expected that an attack based on the evaluation of the contour integral—up to this point the work can usually be carried, except for possible minor modifications—will require essentially novel methods and a most searching examination of the integrand and its behavior on or near the natural boundary.[1] This is exactly what Hardy, with Ramanujan and Littlewood, and making use of an independent investigation by Weyl, has succeeded in doing.[2]

[1] Compare with this the simplicity of the problem of finding the number of *compositions* of n. The generating function $x^s\cdot(x-1)^{-s}$ has $x=1$ as sole singularity in the finite part of the plane.

[2] It is true that Hardy and Littlewood arrange their proofs in the manner that they start out from an arbitrary n (the number to be partitioned) which is then held fixed, so that the number of summands is limited ($\leqq n$) for any mode of partition. Therefore, for the given n, the gener-

11. As the authors insist, there is nothing novel in their plan of attack up to the point of evaluating the integral, or, to be accurate, of determining its order of magnitude as a function of n for large values of n. To quote again Landau, an enthusiastic admirer of the English mathematicians and possibly the world's foremost authority on the subject of the analytic theory of numbers: *The attempt to use Cauchy's integral seemed almost "schülerhaft,"* in view of the obstacles which were to be expected along this road. The work is evidence enough that genius of the highest order and most dogged perseverance were required to overcome these difficulties. Hardy tells us that in the treatment of Waring's problem Littlewood and he were at one point held up for two years, and that they finally succeeded only by linking up their work with the independent investigation by Weyl mentioned before.

Hardy's and Littlewood's later work which is still in full progress tends to smooth off somewhat the roughest and most painful bumps of their first trail. That the work is in its final shape seems hardly probable. Already parts have been simplified by Landau,[1] Weyl,[1] Ostrowski.[1] Besides, Landau[2] and Siegel[2] have extended Hardy's and Littlewood's results to more general problems. The papers in which the theory is developed are scattered in many journals, as is seen from the literature list. The rudiments of the theory appear to go back to at least the Cambridge Mathematical Congress of 1912. For this, compare two long papers by Hardy and Littlewood.[3] The influence of Hardy on Ramanujan and the influence of Ramanujan on Hardy can be traced from Hardy's very appreciative article in honor of the memory of the Indian Mathematician.[4] The first papers dealing with the theory from the standpoint of analytic functions of a complex variable were (**9**), (**10**), (**11**), (**12**), (**13**). Before the theory was farther developed, Ramanujan died (1920). The most delicate parts of the new theory were worked out by Hardy and Littlewood in joint papers.

Up to the present time co-workers in the new field seem to comprise, as far as can be judged by the articles printed, German mathematicians exclusively (Weyl, Landau, Ostrowski, Siegel). Before sketching in greater detail the contents of one of the papers (**13**), we quote two more passages from Landau's

ating function might have been chosen as $f_1(x) = [(1 - x^1)(1 - x^2) \cdots (1 - x^n)]^{-1}$, instead of allowing for an infinite product, and the singularities (on the circumference of the unit circle) would now be poles (and nowhere dense). It may therefore seem unnecessary to consider generating functions with the unit circle as natural boundary; but, on the other hand, the set of singular points of $f_1(x)$ depends on n, while the set of singular points of $f(x)$ is independent of n. It is likely that the apparently simpler case is really the harder one to treat, in somewhat the same sense in which it is ordinarily harder to deal with a large number of terms of a series than to deal with the infinite series and harder to deal with a finite summation than with a definite integral. In the same way, in problem VI for a given n we might think of choosing the rational function $(1 + x^{1^k} + x^{2^k} + \cdots + x^{n^k})^s$, or a still simpler function, instead of the transcendental function $(1 + x^1 + x^{2^k} + \cdots \text{ in inf.})^s$.

[1] (**25**), (**26**), (**30**), (**31**), (**27**).

[2] (**26**), (**28**).

[3] Hardy and Littlewood, "Some problems of Diophantine approximation," *Acta Mathematica*, vol. 37, 1914, pp. 155–191 and pp. 192–239.

[4] *Proceedings of the London Mathematical Society*, 2d ser., vol. 19 (1921), xl–lviii. See also this MONTHLY (*1920*, 316, 338; *1921*, 136, 224, 458).

excellent report,[1] already referred to: *Etwas ganz Gewaltiges ist es, was meine Englischen Freunde geleistet haben*, and: *Ich freue mich, erlebt zu haben, dass die additive Zahlentheorie Methoden erhalten hat.*

(*To be concluded in the next issue.*)

A CONTRIBUTION OF LEIBNIZ TO THE HISTORY OF COMPLEX NUMBERS.[2]

By R. B. McCLENON, Grinnell College.

One of the most important and fascinating chapters in the history of mathematics is the development of the concept of complex numbers. Certain parts of this development have not yet been adequately treated by writers on the history of mathematics; and among these is to be mentioned the work of Leibniz.

It may be worth while to recall that neither the Hindu nor the Arabian algebraists, nor the medieval Europeans, had recognized any possibility of attaching a meaning to a square root of a negative number; indeed it was only the exceptional writer who recognized even *negative* roots of equations.[3] In the sixteenth century, Tartaglia and Cardan, in the formula for the roots of the cubic $x^3 + ax = b$, viz.,

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}},$$

noticed that in case $(b^2/4) + (a^3/27)$ were negative, the value of x would involve an "impossible" expression; and accordingly this case came to be known as the "irreducible case," a term which persists down to the present time. Vieta (1540–1603), the greatest algebraist of his time, contented himself with working out a trigonometric solution for the cubic in this case.[4] Descartes, in connection with his "rule of signs," mentioned the existence of imaginary roots in an algebraic equation, but did not enter upon any discussion of them.[5]

It is now almost exactly 250 years since Leibniz, then a young man of 25, first entered upon the serious study of the possibility of getting some clear meaning out of these so-called "impossible" quantities. The inspiration for this work came to him through the study of Bombelli's *Algebra*, a standard work which had been published at Bologna in 1572 and reprinted in 1579. Leibniz was not at all satisfied with Bombelli's discussion of the "Cardan" formula for the

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[1] (24).

[2] Read before the Iowa Section of the Association, April 27, 1923.

[3] For example, Leonardo of Pisa. *Scritti* (ed. Boncompagni), Rome, 1857, II, pp. 239, 243.

[4] Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert*, Leipzig, 1903, pp. 91, 117; Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. II, 2d ed., Leipzig, 1900, p. 636. Vieta's construction is given in detail in Montucla, *Histoire des Mathématiques*, vol. I, 2d ed., Paris, 1799, p. 605 ff.

[5] Cantor, *loc. cit.*, p. 795.

solution of the cubic equation, especially in the irreducible case. In a letter to Huygens,[1] he expresses his dissatisfaction with Bombelli[2] for not accepting Cardan's formula as adequate in this case; and proceeds to make these three assertions: (1) that Cardan's formula is universally valid, (2) that by means of this formula every cubic equation can be solved, and (3) that roots of all even degrees can be formed which contain imaginaries and yet which are themselves real. As an example of this last, Leibniz mentions that

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}. \tag{1}$$

He also says in this same letter that he has found "a method for extracting, either exactly or approximately, the roots of binomials where imaginaries enter." [3] In reply to this communication, Huygens expresses his astonishment at the relation (1) in these words: "The remark which you make concerning roots that can not be extracted, and containing imaginary quantities which when added together give none the less a real quantity, is surprising and entirely new. One would never have believed that $\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}$ would make $\sqrt{6}$, and there is something hidden in this which is incomprehensible to us." [4]

Leibniz evidently spent considerable time and effort on the question of the meaning of imaginary expressions, and the possibility of securing reliable results by applying to them the ordinary laws of algebra; for Gerhardt found among Leibniz's papers a discussion of the solution of algebraic equations which, although undated, bears every evidence of having been written at about this time (1675).[5] It is published in the *Briefwechsel*, pp. 550–564, and although it is one of the first significant documents in the history of complex numbers, it has not hitherto, so far as I know, been described by historians of mathematics.[6] A rather full description of this paper will accordingly be worth while; and not alone because it has historical importance, but also because the clear-cut way in which Leibniz presents many of his points offers valuable suggestions to the teacher of the present day.

After stating the condition under which a quadratic equation will have real roots, Leibniz continues, "But if now a simple, that is, a linear equation, is multiplied by a quadratic, a cubic equation will result, which will have three

[1] *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern*, ed. C. J. Gerhardt, vol. 1, Berlin, 1899, pp. 547, 548.

[2] This dissatisfaction may perhaps not have been altogether justified, as Bombelli in reality shows considerable skill in handling imaginary expressions. Cf. his *L'Algebra, parte maggiore dell'Arithmetica divisa in tre libri*, Bologna, 1572, pp. 292–293.

[3] *Loc. cit.*, p. 549.

[4] *Loc. cit.*, p. 566.

[5] Gerhardt gives the date 1673; but the editors of Huygens's Works think 1675 more probable. The difference is of no moment in this connexion.

[6] Cantor apparently refers to it (*Geschichte*, vol. III, p. 105 (1894 ed.)) but gives no detailed information. He undervalues the significance of equation (1) above when he says: "Leibniz's formula was to be sure a striking example, but it did not bring the question of its real meaning a single step farther." The obvious advance shown in Leibniz's paper, summarized in this article, shows that Cantor's dictum is in this case too severe.

real roots if the quadratic is possible, or two imaginary roots and only one real one if the quadratic is impossible." He then points out that it is exactly in the case where all three roots of the cubic are real that the difficulty in the use of Cardan's formula lies: the roots of $y^3 + qy - r = 0$ being

$$y = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}.$$

"How can it be," he says, "that a real quantity, a root of the proposed equation, is expressed by the intervention of an imaginary? For this is the remarkable thing, that, as calculation shows, such an imaginary quantity is only observed to enter those cubic equations that have no imaginary root, all their roots being real or possible, as has been shown by trisection of an angle, by Albert Girard and others.[1] * * * This difficulty has been too much for all writers on algebra up to the present, and they have all said that in this case Cardan's rules fail."

Realizing clearly, then, the nature and difficulty of the problem, involving as its solution did a decisive step in advance of all his predecessors, Leibniz set to work to get to the bottom of the matter. He was led to the solution of the problem by an analogy in a similar situation.[2] "It will be useful to mention how my mind was led to the solution of this problem. I once came upon two equations of this kind: $x^2 + y^2 = b$, $xy = c$, whence $x^2 = \frac{c^2}{y^2}$, and so $\frac{c^2}{y^2} + y^2 = b$ and $y^4 - by^2 + c^2 = 0$, or $y^2 = \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}$, $y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}}$. Substituting therefore this value of y^2 in $x^2 + y^2 = b$, I wrote $x^2 - \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} = 0$, or $x = \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}$. But c was greater than b, and therefore $\sqrt{\frac{b^2}{4} - c^2}$ was an imaginary quantity. However, I knew otherwise that the sum of the unknowns $x + y$ was a real quantity and equal to a certain line d, which puzzled me greatly, for inasmuch as I had deduced from the preceding calculation that $d = x + y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}$, I did not understand how such a quantity could be real, when imaginary or impossible numbers were used to express it. I therefore began to retrace the steps of my calculation, suspecting an error; but in vain, for the result was always the same. At length it occurred to me to try this operation: put

$$d = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}} = A + B;$$

[1] Girard, *Invention nouvelle en l'Algebre*, Amsterdam, 1629. Vieta had given the solution by trigonometry earlier than Girard. (See note 4, p. 369.)

[2] Leibniz, *loc. cit.*, pp. 553 ff. Leibniz uses the sign ⌐¬ for equality, but otherwise his symbols are as here given. The sentences in quotation marks are literal translations from the Latin original, and all omissions in the midst of a quotation are indicated by asterisks.

hence, squaring both sides,

$$d^2 = A^2 + B^2 + 2AB = \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} + \frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2} + 2c. \text{ * * *}$$

Therefore $d^2 = b + 2c$, and $d = \sqrt{b + 2c}$. Therefore, equating the two values of d,

$$\sqrt{b + 2c} = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} + \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}.$$

If we put $b = 2$ and c also $= 2$, there results

$$\sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}.$$

I do not remember to have noted a more singular and paradoxical fact in all analysis; for I think I am the first one to have reduced irrational roots, imaginary in form, to real values without extracting them."

Thus Leibniz was led to what he called a *sixth arithmetical operation*, viz., the reduction of imaginary expressions to real form. He then proceeds to apply this operation to the Cardan form of the roots of a cubic equation. And first, he extends the principle of the preceding work with square roots to cube roots, as follows:

"Let $2b$ be a certain quantity: it can be written also in this way: $b + \sqrt{-ac} + b - \sqrt{-ac}$. For although $\sqrt{-ac}$ is an imaginary quantity, yet the sum is none the less real, since the imaginaries are destroyed. Let this formula be divided into two parts, the binomial $b + \sqrt{-ac}$ and the 'apotome'[1] $b - \sqrt{-ac}$, and let us investigate the cube of each separately: the cube of $b + \sqrt{-ac}$ will be

$$b^3 - ac\sqrt{-ac} - 3bac + 3b^2\sqrt{-ac},$$

and the cube of $b - \sqrt{-ac}$ will be

$$b^3 + ac\sqrt{-ac} - 3bac - 3b^2\sqrt{-ac},$$

and therefore

$$\sqrt[3]{\begin{matrix} b^3 - ac\sqrt{-ac} \\ -3bac + 3b^2\sqrt{-ac} \end{matrix}} + \sqrt[3]{\begin{matrix} b^3 + ac\sqrt{-ac} \\ -3bac - 3b^2\sqrt{-ac} \end{matrix}},$$

or

$$\sqrt[3]{\begin{matrix} b^3 \\ -3bac \end{matrix} + \sqrt{\begin{matrix} -a^3c^3 \\ +6a^2c^2b^2 \\ -9b^4ac \end{matrix}}} + \sqrt[3]{\begin{matrix} b^3 \\ -3bac \end{matrix} - \sqrt{\begin{matrix} -a^3c^3 \\ +6a^2c^2b^2 \\ -9b^4ac \end{matrix}}} = 2b,$$

or $b + \sqrt{-ac} + b - \sqrt{-ac}$.

"But if now from a binomial of this kind the cube root can always be extracted, as it can from this one, then certainly the imaginaries can always be

[1] This is the designation used by Euclid in Book X of the *Elements*. (Heath's edition, vol. III, pp. 5–7.) Cf. Heath, *History of Greek Mathematics*, Oxford, 1921, vol. I, p. 407.

removed from a binomial and an 'apotome' when they are joined together. But since it can not always be extracted from a given expression in the form

$$\sqrt[3]{\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}},$$

such as cubic equations give, that is, since the given quantity $r/2$ can not always be separated into two, $b^3 - 3bac$, nor the given quantity $(r^2/4) - (q^3/27)$ into three, $-a^3c^3 + 6a^2c^2b^2 - 9b^4ac$, without another equation, equally as difficult as the given one, therefore it happens that we can not always eliminate the imaginaries from real quantities.

"But it will be useful to give examples in rational numbers. Take the equation, which also Albert Girard used:[1] $x^3 - 13x - 12 = 0$, whose true root is 4. From the formulas of Scipio Ferro or Cardan,

$$x = \sqrt[3]{6+\sqrt{\frac{-1225}{27}}}+\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}.$$

I will prove that this expression is correct and real, and must be admitted. Put $x = 2 + \sqrt{-\frac{1}{3}} + 2 - \sqrt{-\frac{1}{3}}$, and certainly x will be equal to 4, as the equation postulated. Now let us see if the Cardan formula can be derived from this. Certainly by cubing and applying the above formula $b + \sqrt{-ac} + b - \sqrt{-ac}$ to this, making $b = 2$, and $ac = \frac{1}{3}$, we shall have for the cube of $2 + \sqrt{-\frac{1}{3}}$ this formula:[2]

$$\begin{matrix} +8 \\ -3,2,\frac{1}{3} \end{matrix} = -2+\sqrt{\begin{cases} -\frac{1}{27} \\ +6,\frac{1}{9},\ 4=\frac{72}{27} \\ -9,16,\frac{1}{3}=\frac{-1296}{27} \end{cases}}$$

or, adding up, $6+\sqrt{\frac{-1225}{27}}$. In the same way the cube of $2 - \sqrt{-\frac{1}{3}}$ will be $6-\sqrt{\frac{-1225}{27}}$, and hence $\sqrt[3]{6+\sqrt{\frac{-1225}{27}}}$ will be $2 + \sqrt{-\frac{1}{3}}$ and $\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}$ will be $2 - \sqrt{-\frac{1}{3}}$ and, by joining the binomial to the 'apotome,' x or $\sqrt[3]{6+\sqrt{\frac{-1225}{27}}}+\sqrt[3]{6-\sqrt{\frac{-1225}{27}}}$ will be the same as $2 + \sqrt{-\frac{1}{3}} + 2 - \sqrt{-\frac{1}{3}}$, that is, will be 4, as was proposed to show."

Leibniz adds an example where a negative number (-6) is a root of the

[1] *Loc. cit.*

[2] Leibniz, it will be observed, here uses commas to indicate multiplication; he later introduced the dot which has been universally adopted. Cf. Tropfke, *Geschichte der Elementarmathematik*, vol. 2 (2d ed.), Berlin and Leipzig, 1921, p. 24.

cubic, $x^3 - 48x - 72 = 0$, and establishes the fact that

$$\sqrt[3]{36 + \sqrt{-2800}} + \sqrt[3]{36 - \sqrt{-2800}} = -6.$$

He finally takes the bull by the horns, substitutes in the cubic equation $x^3 - qx - r = 0$ the expression for x given by the Cardan formula, and shows by actually carrying out the algebraic reductions that the equation is thus satisfied.

The rest of the memoir is devoted to a discussion of the great difficulty of extending the methods of solution to the 5th, 6th, and higher degree equations, with emphasis upon the necessity of doing this. The concluding sentences are as follows:[1] "For this evil I have found a remedy and obtained a method, by which without experimentation the roots of such binomials can be extracted, imaginaries being no hindrance, and not only in the case of cubics but also in higher equations. This invention rests upon a certain peculiarity which I will explain later. Now I will add certain rules derived from the consideration of irrationals (although no mention is made of irrationals), by which a rational root can easily be extracted from them."

Here the manuscript breaks off; no doubt Leibniz became convinced that he could not carry his "method" as far as he had at first supposed, and thus the essay was left unfinished. But the influence of this work of Leibniz is seen in the writings of Tschirnhausen on the one hand and of John Bernoulli on the other, each of whom received stimulation and valuable assistance from Leibniz in the field of algebra. Thus this particular memoir on complex numbers, although remaining unpublished for two centuries, is an interesting and important document in the history of mathematics.

VECTOR ANALYSIS OF A SURFACE.

By J. B. REYNOLDS, Lehigh University.

A variable vector $\mathbf{r}$ depending upon two independent scalars u and v may be taken as defining a surface. If v is constant, the vector $\mathbf{r}$ defines one of a family of curves lying on the surface called u-lines, while if u is constant, we have one of the family of v-lines lying upon the surface. If there is a relation $\phi(u, v) = 0$, we have a general curve upon the surface.

Now $\mathbf{t} = (d\mathbf{r}/du)$ is a vector parallel to a tangent to the general curve, and $\mathbf{t}_u = (\partial\mathbf{r}/\partial u)$ and $\mathbf{t}_v = (\partial\mathbf{r}/\partial v)$ are vectors parallel, respectively, to tangents to the u- and v-lines.

The vector $\mathbf{n}'' = \mathbf{t}_u \times \mathbf{t}_v$ is parallel to the surface normal, while the vector $d\mathbf{t}_1/du$, $\mathbf{t}_1$ being a unit vector parallel to the tangent, is parallel to the principal normal to the curve, and lies in the osculatory plane

Again, $\mathbf{n}' = \mathbf{t}' \times \mathbf{t}$ is a vector parallel to the binormal, $\mathbf{t}'$ being $d\mathbf{t}/du$.

[1] *Briefwechsel*, p. 564.

Considering v as a function of u, and calling $p = (dv/du)$, we have

$$t = \frac{dr}{du} = t_u + pt_v$$

in which t is parallel to the tangent to the curve and will be defined for a chosen value of p.

Now if ω is the angle between the principal normal to the curve and the normal to the surface at any point, we have, using the notation above,

$$n \cdot n'' = n_0 n_0'' \cos \omega,$$

the subscript zero indicating the absolute value of the vector, or

$$\frac{dt_1}{du} \cdot (t_u \times t_v) = \left(\frac{dt_1}{du}\right)_0 (t_u \times t_v)_0 \cos \omega$$

But, R being the radius of curvature of the curve, we have

$$R = \frac{t_0}{\left(\frac{dt_1}{du}\right)_0},$$

so we may write

$$\frac{\cos \omega}{R} = \frac{\frac{dt_1}{du} \cdot (t_u \times t_v)}{t_0 (t_u \times t_v)_0}. \tag{1}$$

If $\omega = 0$ or π, R becomes the radius of curvature ρ of a normal section of the surface, giving $R = \pm \rho \cos \omega$, which is Meusnier's theorem.

We have, therefore, for a normal section

$$\frac{1}{\rho} = \frac{\left(\frac{dt_1}{du}\right) \cdot (t_u \times t_v)}{t_0 (t_u \times t_v)_0}. \tag{2}$$

From $t = t_u + pt_v$ it follows that $t \times t_v = t_u \times t_v$ since $t_v \times t_v = 0$ so that

$$\left(\frac{dt_1}{du}\right) \cdot (t_u \times t_v) = \frac{dt_1}{du} \cdot (t \times t_v) = \left(\frac{dt_1}{du} \times t\right) \cdot t_v = Sn' \cdot t_v$$

because $(dt_1/du) \times t$ is parallel to the binormal.

To determine the value of the scalar multiplier S, we note that our assumption is

$$\frac{dt_1}{du} \times t = Sn' = S(t' \times t).$$

Now

$$t_1 = \frac{t}{t_0} = \frac{t}{\sqrt{t \cdot t}},$$

therefore

$$\frac{dt_1}{du} = \frac{(t\cdot t)t' - t(t\cdot t')}{(t\cdot t)^{3/2}},$$

whence

$$\frac{dt_1}{du}\times t = \frac{(t\cdot t)(t'\times t) - 0}{(t\cdot t)^{3/2}} = \frac{t'\times t}{t_0},$$

whence $S = 1/t_0$, giving

$$\frac{1}{\rho} = \frac{1}{t_0^2}\frac{n'\cdot t_v}{(t_u\times t_v)_0} = \frac{(t'\times t)\cdot t_v}{t_0^2(t_u\times t_v)_0}. \tag{3}$$

Since the direction of the curve at a point is determined by a constant value of p, we have, from $t = t_u + pt_v$,

$$t' = t_{uu} + 2pt_{uv} + p^2t_{vv}$$

so that

$$(t'\times t)\cdot t_v = \{t_{uu} + 2pt_{uv} + p^2t_{vv}\}\cdot(t_u\times t_v),$$

whence

$$\frac{1}{\rho} = \frac{\{t_{uu} + 2pt_{uv} + p^2t_{vv}\}\cdot n''}{\{t_u\cdot t_u + 2pt_u\cdot t_v + p^2t_v\cdot t_v\}n_0''}. \tag{4}$$

To find the maximum and minimum values of ρ we put $(\partial/\partial p)(1/\rho) = 0$, resulting in the equation

$$(t_u\cdot t_vt_{vv} - t_v\cdot t_vt_{uv})\cdot n''p^2 + (t_u\cdot t_ut_{vv} - t_v\cdot t_vt_{uu})\cdot n''p + (t_u\cdot t_ut_{uv} - t_u\cdot t_vt_{uu})\cdot n'' = 0.$$

If p and p' are the roots of this equation,

$$p + p' = \frac{(t_v\cdot t_vt_{uu} - t_u\cdot t_ut_{vv})\cdot n''}{(t_u\cdot t_vt_{vv} - t_v\cdot t_vt_{uv})\cdot n''}$$

and

$$pp' = \frac{(t_u\cdot t_ut_{uv} - t_u\cdot t_vt_{uu})\cdot n''}{(t_u\cdot t_vt_{vv} - t_v\cdot t_vt_{uv})\cdot n''}, \tag{5}$$

whence

$$t_v\cdot t_vpp' + t_v\cdot t_u(p + p') + t_u\cdot t_u = 0 \tag{6}$$

or

$$(t_u + pt_v)\cdot(t_u + p't_v) = 0.$$

That is, sections of maximum and minimum curvature are perpendicular to each other.

We have then, if ρ_1 and ρ_2 are the principal radii of curvature, simplifying (4) by (6) and the expression for $p + p'$,

$$\frac{1}{\rho_1} = \frac{(t_{uv} + pt_{vv})\cdot n''}{(t_u\cdot t_v + pt_v\cdot t_v)n_0''} \quad \text{and} \quad \frac{1}{\rho_2} = \frac{(t_{uv} + p't_{vv})\cdot n''}{(t_u\cdot t_v + p't_v\cdot t_v)n_0''}, \tag{7}$$

and adding, we obtain for the mean curvature at any point by means of equations

(5) and (6):

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{(2t_u\cdot t_v t_{uv}-t_v\cdot t_v t_{uu}-t_u\cdot t_u t_{vv})\cdot n''}{n_0''[(t_u\cdot t_v)^2-(t_u\cdot t_u)(t_v\cdot t_v)]}.$$

But since

$$\begin{aligned} n_0''^2 &= (t_u\times t_v)\cdot(t_u\times t_v)=t_u\cdot t_v\times(t_u\times t_v)\\ &= t_u\cdot[t_u(t_v\cdot t_v)-t_v(t_v\cdot t_u)]\\ &= (t_u\cdot t_u)(t_v\cdot t_v)-(t_u\cdot t_v)^2, \end{aligned}$$

we have

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{(t_v\cdot t_v t_{uu}+t_u\cdot t_u t_{vv}-2t_u\cdot t_v t_{uv})\cdot n''}{n_0''^3}. \tag{8}$$

In like manner we find for the total curvature

$$\frac{1}{\rho_1\rho_2}=\frac{(t_{uu}\cdot n'')(t_{vv}\cdot n'')-(t_{uv}\cdot n'')^2}{n_0''^4}. \tag{9}$$

At umbilics the curvature of all normal sections is the same, so by (4)

$$\frac{t_{uu}\cdot n''}{t_u\cdot t_u}=\frac{t_{uv}\cdot n''}{t_u\cdot t_v}=\frac{t_{vv}\cdot n''}{t_v\cdot t_v} \tag{10}$$

for the determination of such points.

If $p=0$ and $p'=\infty$, we get from (4) for the curvature of principal sections

$$\frac{1}{\rho_1}=\frac{t_{uu}\cdot n''}{(t_u\cdot t_u)n_0''},\qquad \frac{1}{\rho_2}=\frac{t_{vv}\cdot n''}{(t_v\cdot t_v)n_0''}. \tag{11}$$

If the curves of reference are lines of curvature, we must have the conditions that they intersect perpendicularly, and that $p'=\infty$ when $p=0$. Considering the quadratic for p we may write for these conditions

$$t_u\cdot t_v=0 \qquad \text{and} \qquad n''\cdot t_{uv}=0. \tag{12}$$

So, basing our analysis on lines of curvature, we have by (4)

$$\frac{1}{\rho_1}=\frac{(t_{uu}+p^2t_{vv})\cdot n''}{(t_u\cdot t_u+p^2t_v\cdot t_v)n_0''}$$

and

$$\frac{1}{\rho_2}=\frac{(t_{uu}+p'^2t_{vv})\cdot n''}{(t_u\cdot t_u+p'^2t_v\cdot t_v)n_0''},$$

whence, since the sections are perpendicular and

$$pp'=\frac{-t_u\cdot t_u}{t_v\cdot t_v},$$

we find

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{t_{uu}\cdot n''}{t_u\cdot t_u n_0''}+\frac{t_{vv}\cdot n''}{t_v\cdot t_v n_0''}$$
$$=\frac{1}{n_0''}\left(\frac{t_{uu}}{t_u\cdot t_u}+\frac{t_{vv}}{t_v\cdot t_v}\right)\cdot n''.$$

That is, the sum of the curvatures of mutually perpendicular sections is constant.

Again, using lines of curvature as reference lines, in general, by (4)

$$\frac{1}{\rho}=\frac{(t_{uu}+p^2t_{vv})\cdot n''}{(t_u\cdot t_u+p^2t_v\cdot t_v)n_0''}.$$

If θ is the angle between the plane of the normal section in question and the plane of the normal section containing the tangent to the u-line at the point, we have

$$\tan\theta=\frac{(t\times t_u)_0}{t\cdot t_u}$$
$$=\frac{[(t_u+pt_v)\times t_u]_0}{(t_u+pt_v)\cdot t_u}$$
$$=\frac{p(t_v\times t_u)_0}{t_u\cdot t_u}$$
$$=\frac{pn_0''}{t_u\cdot t_u},$$

whence

$$p^2=\frac{(t_u\cdot t_u)^2}{n_0''^2}\tan^2\theta=\frac{t_u\cdot t_u}{t_v\cdot t_v}\tan^2\theta,$$

because $n_0''^2=(t_u\cdot t_u)(t_v\cdot t_v)-(t_u\cdot t_v)^2$ and, in this case, $t_u\cdot t_v=0$, so that we have

$$\frac{1}{\rho}=\frac{t_{uu}\cdot n''}{n_0''t_u\cdot t_u}\cos^2\theta+\frac{t_{vv}\cdot n''}{n_0''t_v\cdot t_v}\sin^2\theta$$
$$=\frac{\cos^2\theta}{\rho_1}+\frac{\sin^2\theta}{\rho_2},$$

which is Euler's Theorem.

Since for elliptic points the curvatures of both principal sections have the same sign, we have by (9) for such points

$$(t_{uu}\cdot n'')(t_{vv}\cdot n'')>(t_{uv}\cdot n'')^2$$

and (13)

$$(t_{uu}\cdot n'')(t_{vv}\cdot n'')\gtreqless(t_{uv}\cdot n'')^2$$

for parabolic and hyperbolic points respectively.

As in the first part of this article, considering the angle ω between the surface normal n and the principal normal n'' of a curve on the surface, we may write

$(n \times n'')_0 = n_0 n_0'' \sin \omega$ whence

$$\sin \omega = \frac{\left[\frac{dt_1}{du} \times (t_v \times t_u)\right]_0}{\left(\frac{dt_1}{du}\right)_0 (t_v \times t_u)_0} = \frac{R\left\{\frac{dt_1}{du} \times (t_v \times t_u)\right\}_0}{t_0(t_u \times t_v)_0}.$$

Now $\frac{dt_1}{du} =$ is parallel to $t \times (t \times t')$ whence

$$\frac{dt_1}{du} = s\{t \times (t \times t')\},$$

s being a scalar. To determine s, we have, from $t_1 = \frac{t}{\sqrt{t \cdot t}}$,

$$\frac{dt_1}{du} = \frac{(t \cdot t)t' - t(t \cdot t')}{(t \cdot t)^{3/2}}$$

and

$$t \times (t \times t') = t(t \cdot t') - t'(t \cdot t),$$

that

$$s = -\frac{1}{(t \cdot t)^{3/2}} = -\frac{1}{t_0^3}$$

and finally

$$\frac{\sin \omega}{R} = \frac{[(t_v \times t_u) \times \{t \times (t \times t')\}]_0}{t_0^4 (t_u \times t_v)_0}. \tag{14}$$

If $\omega = 90°$ we have for R the radius of the projection of the curve in the tangent plane at the point; hence, if R_t is the tangential or geodesic radius of curvature, we may write

$$\frac{1}{R_t} = \frac{[(t_v \times t_u) \times \{t \times (t \times t')\}]_0}{t_0^4 (t_u \times t_v)_0}. \tag{15}$$

By two applications of the vector formula $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ to this we get

$$\frac{1}{R_t} = \frac{\{t_u[(t_v \cdot t)(t \cdot t') - (t_v \cdot t')(t \cdot t)] - t_v[(t_u \cdot t)(t \cdot t') - (t_u \cdot t')(t \cdot t)]\}_0}{t_0^4 (t_u \times t_v)_0}.$$

Now, referring to lines of curvature as reference lines, in which case $t_u \cdot t_v = 0$, we find, since $t = t_u + p t_v$,

$$\frac{1}{R_t} = \frac{[p(t_v \cdot t_v)(t_u \cdot t') - (t_u \cdot t_u)(t_v \cdot t')] \sqrt{t_u \cdot t_u + p^2 t_v \cdot t_v}}{t_0^4 (t_u \times t_v)_0}. \tag{16}$$

To find the equation determining geodesics, we have that the normal to a

geodesic coincides with the normal to the surface giving $\omega = 0$ whence

$$\frac{1}{R_t} = \frac{\sin \omega}{R} = 0$$

so that for geodesics we have

$$(\boldsymbol{t}_u \cdot \boldsymbol{t}_u)(\boldsymbol{t}_v \cdot \boldsymbol{t}') = p(\boldsymbol{t}_v \cdot \boldsymbol{t}_v)(\boldsymbol{t}_u \cdot \boldsymbol{t}'). \tag{17}$$

This results in a second order third degree differential equation to determine such curves, because in general

$$\boldsymbol{t}' = \boldsymbol{t}_{uu} + 2p\boldsymbol{t}_{uv} + p^2\boldsymbol{t}_{vv} + \frac{dp}{du}\boldsymbol{t}_v.$$

Application. Consider the torus whose rectangular equation is

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(x^2 + y^2)$$

generated by revolving the circle $(x - c)^2 + z^2 = a^2$ about the z-axis.

Letting u be the angle between the radius to any point on a circular section and the xy-plane, and v the angle between the plane of the section and the xz-plane, we may write for a vector equation of this surface

$$\boldsymbol{r} = (c + a\cos u)\cos v\boldsymbol{i} + (c + a\cos u)\sin v\boldsymbol{j} + a\sin u\boldsymbol{k},$$

$\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ being mutually perpendicular unit vectors parallel to the x-, y-, and z-axes respectively.

From this we obtain

$$\begin{aligned}
\boldsymbol{t}_u &= -a\sin u\cos v\boldsymbol{i} - a\sin u\sin v\boldsymbol{j} + a\cos u\boldsymbol{k},\\
\boldsymbol{t}_v &= -(c + a\cos u)\sin v\boldsymbol{i} + (c + a\cos u)\cos v\boldsymbol{j},\\
\boldsymbol{t}_{uu} &= -a\cos u\cos v\boldsymbol{i} - a\cos u\sin v\boldsymbol{j} - a\sin u\boldsymbol{k},\\
\boldsymbol{t}_{vv} &= -(c + a\cos u)\cos v\boldsymbol{i} - (c + a\cos u)\sin v\boldsymbol{j},\\
\boldsymbol{t}_{uv} &= a\sin u\sin v\boldsymbol{i} - a\sin u\cos v\boldsymbol{j},\\
\boldsymbol{n}'' &= \boldsymbol{t}_u \times \boldsymbol{t}_v = -a(c + a\cos u)\cos u\cos v\boldsymbol{i} - a(c + a\cos u)\cos u\sin v\boldsymbol{j}\\
&\qquad - a(c + a\cos u)\sin u\boldsymbol{k}.
\end{aligned}$$

To ascertain by (12) whether our lines of reference are lines of curvature, we have

$$\boldsymbol{t}_u \cdot \boldsymbol{t}_v = a(c + a\cos u)\sin u\sin v\cos v - (c + a\cos u)\sin u\sin v\cos v = 0,$$

$$\begin{aligned}
\boldsymbol{n}'' \cdot \boldsymbol{t}_{uv} &= -a^2(c + a\cos u)\sin u\sin v\cos u\cos v\\
&\qquad + a^2(c + a\cos u)\sin u\sin v\cos u\cos v = 0,
\end{aligned}$$

so that these conditions are met.

To determine the total curvature by (9) we have

$$\begin{aligned}
\boldsymbol{t}_{uu} \cdot \boldsymbol{n}'' &= a^2(c + a\cos u); & \boldsymbol{t}_{vv} \cdot \boldsymbol{n}'' &= a(c + a\cos u)^2\cos u\\
\boldsymbol{t}_{uv} \cdot \boldsymbol{n}'' &= 0; & \boldsymbol{n}_0''^2 &= a^2(c + a\cos u)^2
\end{aligned}$$

giving the expression

$$\frac{1}{\rho_1\rho_2} = \frac{\cos u}{a(c + a\cos u)}$$

so that, according to (13), if $c > a$ we have elliptic points, $-(\pi/2) < u < (\pi/2)$; parabolic, $u = \pm(\pi/2)$; and hyperbolic, $(\pi/2) < u < (3/2)\pi$.

If $c < a$ we have elliptic points for $-(\pi/2) < u < (\pi/2)$ and for $\pi - \cos^{-1}(c/a) < u < \pi + \cos^{-1}(c/a)$; parabolic, $u = \pm(\pi/2)$; infinite curvature, $u = \pi \pm \cos^{-1}(c/a)$; and hyperbolic points for $(\pi/2) < u < \pi - \cos^{-1}(c/a)$ and for $\pi + \cos^{-1}(c/a) < u < (3\pi/2)$.

By (10) for umbilics we have

$$(c + a\cos u)^3 = a\cos u(c + a\cos u)^2$$

so that if $c > a$ there are no umbilics, while if $c < a$ there are umbilics at $u = \pi + \cos^{-1}(c/a)$ which are two points on the z-axis. If $c = 0$ every point is an umbilic, and the surface is a sphere.

By (11) we get for the curvature of the principal sections

$$\frac{1}{\rho_1} = \frac{\boldsymbol{t}_{uu}\cdot\boldsymbol{n}''}{(\boldsymbol{t}_u\cdot\boldsymbol{t}_u)\boldsymbol{n}_0''} = \frac{a^2(c + a\cos u)}{a^3(c + a\cos u)} = \frac{1}{a},$$

$$\frac{1}{\rho_2} = \frac{\boldsymbol{t}_{vv}\cdot\boldsymbol{n}''}{(\boldsymbol{t}_v\cdot\boldsymbol{t}_v)\boldsymbol{n}_0''} = \frac{a(c + a\cos u)^2\cos u}{a(c + a\cos u)^3} = \frac{\cos u}{c + a\cos u},$$

so the one principal section is a circle of radius a; the radius of the other varying from $a \pm c$ to ∞, being ∞ for $u = (\pi/2)$ and zero for $a = \pi - \cos^{-1}(c/a)$ when $c < a$.

To determine geodesics on this surface we have from

$$\boldsymbol{t}' = \boldsymbol{t}_{uv} + 2p\boldsymbol{t}_{uv} + p\boldsymbol{t}_{uv} + \frac{dp}{du}\boldsymbol{t}_v$$

that

$$\boldsymbol{t}'\cdot\boldsymbol{t}_u = ap^2(c + a\cos u)\sin u,$$

$$\boldsymbol{t}'\cdot\boldsymbol{t}_v = (c + a\cos u)[(c + a\cos u)\frac{dp}{du} - 2pa\sin u] + (c + a\cos u)\frac{dp}{du}$$

and the resulting differential equation is

$$a(c + a\cos u)\frac{dp}{du} - 2pa^2\sin u = (c + a\cos u)^2\sin up^3.$$

Two particular solutions are evident.

If $p = 0$, $(dp/du) = 0$, that is $v = c_1$, showing that the u-lines are geodesics.

If $p = \infty$ and $u = 0$ or π, we have two particular v-lines, circles of radii $c \pm a$ as geodesics.

If we let $(c + a\cos u)p = a\tan\theta$, the differential equation becomes

$$\cot\theta\, d\theta - \frac{a\sin u}{c + a\cos u}\, du = 0,$$

whence $(c + a\cos u)\sin\theta = c_1$, or

$$p = \frac{dv}{du} = \frac{a\tan\theta}{c + a\cos u} = \frac{ac_1}{(c + a\cos u)\sqrt{(c + a\cos u)^2 - c_1^2}}$$

or, in general, for geodesics on the torus

$$v = ac_1 \int \frac{du}{\sqrt{(c + a\cos u)^4 - c_1^2(c + a\cos u)^2}} + c_2.$$

If $c_1 = 0$ we have the first solution noted above.

If $c = 0$

$$v = c_1 \int \frac{\sec^2 u\, du}{\sqrt{(a^2 - c_1^2) - c_1^2 \tan^2 u}} = \sin^{-1} \frac{c_1 \tan u}{\sqrt{a^2 - c_1^2}} - \beta$$

or

$$\sin(v + \beta) = \tan\alpha \tan u,$$

where

$$\tan\alpha = \frac{c_1}{\sqrt{a^2 - c_1^2}},$$

giving for geodesics on the sphere

$$r = \frac{a\tan\alpha\cos v}{\sqrt{\tan^2\alpha + \sin^2(v+\beta)}}\, i + \frac{a\tan\alpha\sin v}{\sqrt{\tan^2\alpha + \sin^2(v+\beta)}}\, j + \frac{a\sin(v+\beta)}{\sqrt{\tan^2\alpha + \sin^2(v+\beta)}}\, k,$$

α and β being arbitrary constants. This is a curve lying in the plane

$$z\tan\alpha = x\sin\beta + y\cos\beta$$

and is therefore a great circle of the sphere.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

REPLY.

48 (*1923*, 136). **In what quadratic realms of rationality and for what values of p (which is a prime integer) is the function $\frac{x^p - 1}{x - 1}$ factorable?**

Reply by LOUIS WEISNER, University of Rochester.

The group G of the cyclotomic equation $(x^p - 1)/(x - 1) = 0$, where p is an odd prime, is a regular cyclic group of order $p - 1$, generated by the substitution $s = (\omega\ \omega^2 \cdots \omega^{p-1})$, where ω is a primitive root modulo p. If $\varphi(\epsilon, \epsilon^2, \cdots \epsilon^{p-1})$, where $\epsilon, \epsilon^2, \cdots \epsilon^{p-1}$ denote the roots of the equation, is a function belonging to the subgroup H generated by s^2, the numerical value of φ is of the form $a + b\sqrt{c}$, where a, b and c is not a perfect

square. Thus the equation is reducible in $R(\sqrt{c})$ and in no other quadratic realm, since H is the only subgroup of index 2 in G.

To determine c, consider the function

$$\varphi_1 = \epsilon^{\omega} + \epsilon^{\omega^3} + \cdots + \epsilon^{\omega^{p-2}}$$

which belongs to H and under G takes the second value

$$\varphi_2 = \epsilon + \epsilon^{\omega^2} + \cdots + \epsilon^{\omega^{p-3}}.$$

The numbers φ_1, φ_2 satisfy the equation

$$\varphi^2 - (\varphi_1 + \varphi_2)\varphi + \varphi_1\varphi_2 = 0$$

whose coefficients are rational numbers which we proceed to evaluate. We have at once $\varphi_1 + \varphi_2 = -1$, and [1]

$$\varphi_1\varphi_2 = \sum \epsilon^{\omega^{\alpha}+\omega^{\beta}} \qquad \begin{pmatrix} \alpha = 1, 3, \cdots p-2 \\ \beta = 2, 4, \cdots p-3 \end{pmatrix}.$$

If possible, let $\omega^{\alpha} + \omega^{\beta} \equiv 0 \bmod p$. Then $\omega^{2\alpha} \equiv \omega^{2\beta} \bmod p$, whence $2\alpha \equiv 2\beta \bmod p-1$ and $\alpha \equiv \beta \bmod (p-1)/2$. Since α is odd while β is even, and α, $\beta < p$, this congruence is satisfied only if $\alpha = \beta \pm (p-1)/2$. This requires that $(p-1)/2$ be odd.

(1) Hence if $(p-1)/2$ is even, $\varphi_1\varphi_2$ is a sum of $\frac{1}{4}(p-1)^2$ imaginary pth roots of unity. Since this sum equals a rational number, it must be a multiple of $\epsilon + \epsilon^2 + \cdots + \epsilon^{p-1}$. Thus the $\frac{1}{4}(p-1)^2$ terms, when combined, give $p-1$ distinct terms with the same coefficient, which is therefore $(p-1)/4$. Hence $\varphi_1\varphi_2 = -(p-1)/4$. It follows that

$$\varphi^2 + \varphi - \frac{p-1}{4} = 0,$$

whence

$$\varphi_1, \varphi_2 = \frac{-1 \pm \sqrt{p}}{2}.$$

(2) If $(p-1)/2$ is odd, let us take for β any positive even integer less than p, and $\alpha = \beta \pm (p-1)/2$, the sign being chosen so that α is positive and less than p. Then α is an odd number and

$$\omega^{\alpha} \equiv \omega^{\beta}\omega^{\pm\frac{p-1}{2}} \equiv -\omega^{\beta} \bmod p,$$

so that there are $(p-1)/2$ values of α and β for which $\omega^{\alpha} + \omega^{\beta} \equiv 0 \bmod p$. Therefore $\varphi_1\varphi_2$ is equal to $(p-1)/2$ plus a sum of $(p-1)^2/4 - (p-1)/2$ imaginary pth roots of unity. This sum equals the rational number

$$\frac{p-1}{4} - \frac{1}{2} = \frac{p-3}{4}.$$

[1] For a different evaluation of $\varphi_1\varphi_2$, see Netto, *Substitutionentheorie und ihre Anwendung auf die Algebra* (1882), page 186.

Hence

$$\varphi_1\varphi_2 = \frac{p-1}{2} - \frac{p-3}{4} = \frac{p+1}{4}.$$

Therefore

$$\varphi^2 + \varphi + \frac{p+1}{4} = 0,$$

whence

$$\varphi_1, \varphi_2 = \frac{-1 \pm \sqrt{-p}}{2}.$$

We therefore have the following theorem: *The function* $(x^p - 1)/(x - 1)$, *where* p *is an odd prime, is factorable in* $R(\sqrt{(-1)^{(p-1)/2}p})$, *and is not factorable in any other quadratic realm.*

DISCUSSIONS

I. The Infinite and Imaginary in Algebra and Geometry: A Reply.[1]

By W. L. G. Williams, Cornell University.

"The traditional treatment of imaginary and infinite elements in algebra and geometry," says Professor R. M. Winger in a recent number of this Monthly, "has curiously resisted the reform movement in American text-book writing." What this reform movement is, where it exists, and what text-books its adherents have written, we do not know and until we can examine its products we should hesitate to call it a reform movement.

However, the two questions that Professor Winger raises, the question of the introduction into our elementary college courses in algebra of a number ∞ and the question of considering, in our elementary courses in analytic geometry, imaginary as well as real points, are so different that they require separate examination.

1. Professor Winger points out the service of a line at infinity in rendering the principle of duality universally valid and in preserving a one-to-one correspondence between a figure and its projection. This idea of a line at infinity, which Poncelet introduced into geometry just a century ago, has nevertheless given rise to many misconceptions and errors. One form of the equation of the line at infinity is the one which appears in the article under discussion, viz.,

$$0x + 0y + 1 = 0,$$

and another is

$$1 = 0,$$

equations satisfied by no real or imaginary points. For mathematics as it now exists these two equations are identical and equally meaningless as equations of lines or of any other loci. The second of these equations is evidently quite independent of x and y, but it has been proposed by Professor Winger and

[1] *V.* Infinite and Imaginary Elements in Algebra and Geometry, this Monthly (*1922*, 290).

others to introduce into algebra one or more infinite numbers with the property that their products by zero are different from zero in order to give a meaning to the first of these equations.

Professor Winger is right in believing that if infinite numbers can render a service in analytic geometry they can also do so in algebra, since analytic geometry is an application of algebra. For their use in either we should require a logically constructed theory of the system of numbers obtained by adjoining to the real system or the complex system one or more infinite numbers.

The most important properties of the real and complex number systems are, perhaps, that addition, subtraction, and multiplication in them are always possible and always unique, that the same is true of division when the divisor is different from zero, and that the result is always a number of the system. These properties are not inherent in the nature of a number system, but the systems which do not have them seem so bizarre to all except the professional mathematician and are so unimportant relatively that we find it hard to believe that such systems should have any part in the curriculum of our colleges.

Professor Winger proposes to create a number system in which there shall exist a single infinite number ∞ defined by the relation

$$\frac{k}{0} = \infty, \qquad (k \neq 0).$$

What meaning is to be attached to the words, "k divided by zero"? By the words, "six divided by two," we mean the number x such that $2x = 6$, so that division is the operation inverse to multiplication. In every number system familiar to the present writer in which division has a meaning it is the inverse of multiplication, which is an operation with a definite meaning. If Professor Winger intends to follow the analogy he will say

$$k = 0 \cdot \infty = \infty \cdot 0, \qquad (k \neq 0),$$

but we find it difficult to imagine what this means. It certainly does not mean

$$k = 0 + 0 + 0 + \cdots,$$

for $k \neq 0$. But whatever it may mean we have, for example (operating formally), $0 \cdot \infty = 1$, $0 \cdot \infty = 2$, $\cdots$. In fact, $0 \cdot \infty =$ any number of the system except 0 and possibly ∞.

Such a system of numbers, if it has any uses, is so much more difficult to deal with than quaternions or even linear non-associative algebras that we can hardly consider its study fitting for elementary students.

After laying down the above definition, but without enunciating any other laws of operation, Professor Winger states: "It is then but a step to the theorem that the equation

$$a_0 x^n + \cdots + a_n = 0$$

has r infinite roots if

$$a_0 = a_1 = \cdots = a_{r-1} = 0$$

so that an equation of apparent degree $(n - r)$ may be regarded as an equation of degree n."

Granting that the phrase "apparent degree $(n - r)$" has a meaning, I do not know what it means to say that this equation has r infinite roots, but I am unable to prove that it has one infinite root. In fact there is one simple case in which it has none.

In making this proof I assume that a root of an equation is a number which satisfies it. Consider the equation

$$0 \cdot x + 0 = 0$$

which satisfies all of Professor Winger's conditions. Substituting ∞ for x we have $k + 0 \neq 0$ since by the definition $k \neq 0$. If any coefficient is different from zero the theorem cannot be verified, *e.g.*, consider the equation

$$0 \cdot x^2 + x = 0.$$

This equation is satisfied if and only if $0 \cdot \infty^2 + \infty = 0$, *i.e.*, if and only if $k \cdot 0 + \infty = 0$, *i.e.*, if and only if $(k + 1)\infty = 0$.[1] We do not know the value of k except that it is not 0; if $k = -1$ this necessary and sufficient condition reduces to $0 \cdot \infty = 0$, which is untrue since $0 \cdot \infty = k \neq 0$ by hypothesis. If $k \neq -1$ or 0 we have no means of knowing whether or not the equation is satisfied. Thus we see that it is impossible to verify Professor Winger's theorem even in an extremely simple case.

A useful conception in respect to points at infinity is that there is one and only one on every straight line, and that all straight lines parallel to a given straight line pass through the same point at infinity.[2] Let us see how this fits in with Professor Winger's number system. In his system there exist the following points at infinity: $(c, \infty)(\infty, c)$ and (∞, ∞) where $c \neq \infty$. A point (c, ∞) lies on the line $x + y = 0$ if and only if $c + \infty = 0$ and a point (∞, c) lies on this line if and only if $\infty + c = 0$. Since $c \neq \infty$, no points with the coördinates (∞, c) or (c, ∞) lie on $x + y = 0$. The point (∞, ∞) lies on it if and only if $\infty + \infty = 0$.[3] Whether or not this is true Professor Winger's definition of ∞ does not enable us to say. If, however, $\infty + \infty = 0$, and if $\infty^2 = \infty$ (since formally $\infty^2 = (1/0)^2 = (1/0) = \infty$), then ∞ is a root of the equation $x^2 + x = 0$, and, consequently, this equation has three roots, 0, 1, and ∞. Accordingly, either we have a line (and indeed an infinite number of lines) which contains no point at infinity or we have an equation of the second degree with three roots, so that Professor Winger's system leads us even in the simplest kinds of cases to results inconsistent with the general theorems and postulates whose universality he seeks to maintain.

The author in the incompleteness and vagueness of his statements about his

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[1] Formally, $(k + 1)\infty = (k + 1)\frac{1}{0} = \frac{k + 1}{0} = \infty$, when $k \neq -1$.

[2] See Winger, *loc. cit.*, page 296.

[3] Probably $\infty + \infty = \infty$, for $\infty + \infty = \frac{1}{0} + \frac{1}{0} = \frac{2}{0} = \infty$; possibly $\infty + \infty = \frac{1}{0} + \frac{-1}{0} = \frac{0}{0}$.

number system is but typical of those who talk about a number ∞. Turn to a work on the theory of functions of a real variable like that of Pierpont or de la Vallée Poussin and you find that the greatest care is taken to lay the foundations of the real number system; turn to a work on the complex variable like that of Burkhardt or Pringsheim and you find no vagueness as to this system; quaternions and other unusual systems receive an exact treatment in Dickson's Cambridge Tract; but those who talk of the number ∞ act as if all that was needed was to define it in a formal way.

Since, for the purposes of algebra and analytic geometry there is no logically constructed system in which ordinary real and infinite numbers exist side by side, we must reject not only from our teaching but from all our mathematical work any such "system" until and unless such a system shall be created, and thus avoid the paradoxes which arise in the works of those who attempt to make use of infinite numbers.[1]

A great number of text-books contain such misstatements and misconceptions regarding the so-called infinite that the honest student is bewildered and discouraged by them. These errors and deceptive statements arise from the use of infinite numbers, from imperfect realization of the fact that the infinite regions of different geometries[2] are different, and from a desire to claim more for mathematics than is justified by the facts.

Listen, for example, to Salmon, that most inspiring of romanticists: "Let A and B be both $= 0$, then the intercepts become infinite and the line is altogether situated at an infinite distance from the origin. Now it was proved (Art. 63) that the equation under consideration is equivalent to $0x + 0y + C = 0$, and though it cannot be satisfied by any finite values of the coördinates, it may be by infinite values since the product of nothing by infinity may be finite. * * * * * *; and that the equation of an infinitely distant right line, in Cartesian coördinates, is $0x + 0y + C = 0$. We shall, for shortness, commonly cite the latter equation in the less accurate form $C = 0$."

Another author, whose book is in many respects interesting and excellent, solves the equations

$$y = \pm\left(\frac{b}{a} - \epsilon\right)x$$

with

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

[1] Bôcher: *Bulletin of the American Mathematical Society*, vol. 11 (1904–05), page 134. "Finally there is what may perhaps be called the method of optimism which leads us either wilfully or instinctively to shut our eyes to the possibility of evil. Thus the optimist who treats a problem in algebra or analytic geometry will say, if he stops to reflect on what he is doing: 'I know that I have no right to divide by zero; but there are so many other values which the expression by which I am dividing might have that I will assume that the Evil One has not thrown a zero in my denominator this time.' This method, if a proceeding often unconscious can be called a method, has been of great service in the rapid development of many branches of mathematics, though it may well be doubted whether in a subject as highly developed as is ordinary algebra it has not now survived its usefulness."

[2] Bôcher: Infinite Regions of Various Geometries, *Bulletin of the American Mathematical Society*, vol. 20 (1913–14), page 185.

and finds that each of these lines cuts the hyperbola in points whose x coördinates satisfy the relation

$$x^2 = \frac{1}{\frac{2\epsilon}{ab} - \frac{\epsilon^2}{b^2}}.$$

He then adds: "These two lines then tend as ϵ becomes smaller and smaller to become tangents to the hyperbola, for the points in which either of them meets the hyperbola are at a very great distance and they are on opposite sides of the centre. And it is one of the paradoxes of geometry with which the student is probably acquainted that the points on a line in two opposite directions and at a very great distance tend to become the same point."

This treats geometry as an experimental science in which the physical phenomenon of points moving in opposite directions and yet eventually almost colliding has already startled the young student.

Another author writes: "Thus of the coördinates x, y, z, certainly two and possibly all three are infinite and the point x, y, z is at infinity. We see then that two lines may be so situated that instead of a finite intersection they have their intersection at infinity."

There is not a hint of axiom, definition, postulate; in books of this character infinite points, a line at infinity, circular points at infinity, are as real, as much a part of the physical universe, as thrilling, as Robinson Crusoe to a boy of ten. Salmon would no more have given away the game than would Defoe have printed on the titlepage of his immortal work: "Small boys are hereby warned that no such person as Robinson Crusoe ever existed. Those who think he did, do so at their own risk."

The algebraist who reads these works, the last of which was printed in the present century by one of the world's greatest presses, will hardly cry out in envy of the geometer: "My subject is every whit as good as his; why can't I take over a little of this wonderful thing called infinity and make my subject fascinating too?"

At the top of page 294 Professor Winger says: "For example, the student has learned to expect two solutions of a linear and a quadratic equation even though the roots be coincident or imaginary." If the student has learned to expect any such thing the business of his teacher is to show him that he has been misled, not to try to invent some system of numbers which he will not understand in order apparently to justify a totally false expectation. Neither in the domain of real numbers nor in the domain of complex numbers nor in the system obtained by Professor Winger do we know that

$$\begin{aligned} x^2 - 9y^2 &= 7, \\ x &= 3y + 1 \end{aligned}$$

have two solutions. In the real domain they have but one, in the complex domain they have one and only one, and we have no means of knowing whether

(∞, ∞) is another solution in Professor Winger's domain, but if so it is of very little interest for algebra in the ordinary sense or for analytic geometry.

Professor Winger seems to imply that without taking his point of view it is impossible to tell which of

$$x = 3y + 1,$$
$$x = 3y$$

is an asymptote to

$$x^2 - 9y^2 = 7.$$

It is true as Professor Winger states (page 294) that an asymptote may be defined as the limit of a tangent (if it exist), but the method of treatment of asymptotes which he suggests has nothing to do with limits. It is in various ways easy to lay down a rule of thumb method for finding asymptotes, but modern mathematics has nothing to do with rule of thumb methods. The principles of finding asymptotes, whether we define one as Professor Winger suggests, or as a line whose distance from a point on the curve $\rightarrow 0$ as one or both of its coördinates $\rightarrow \infty$, depend upon a knowledge of limits and of infinity in the sense of all analysts since Cauchy.

With the word "infinity" are bound up some of the most important ideas in mathematics. There are points at infinity in analytic geometry, but their coördinates are not infinite numbers; there is a line at infinity, but its equation is not $0x + 0y + 1 = 0$; there are infinite collections of objects, but the sum of an infinite number of zeros is not a number different from zero. The experience of many generations of mathematicians has shown that the ideas of limit and infinity present important psychological difficulties. A Bernoulli could say: "If there are an infinite number of terms (in an infinite sequence) there must be an infinitieth, for the same reason that if there are ten there must be a tenth." An Euler could say that

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

and the haziness of his notions is not excused by modern developments in divergent series. Though Abel at twenty called divergent series an invention of the devil even his great genius could not bring order out of the chaos, a chaos which was not dispelled until Cauchy and Weierstrass had devoted long lives to reconstructing the foundations of mathematics.

With regard to the teaching of these difficult ideas to students G. H. Hardy in the preface to that excellent book, "A Course of Pure Mathematics," says: "It has been my good fortune during the last eight or nine years to have a share in the teaching of a good many of the ablest candidates for the Mathematical Tripos; and it is very rarely indeed that I have encountered a pupil who could face the simplest problem involving the ideas of infinity, limit or continuity with a vestige of the confidence with which he would deal with questions of a different character and of far greater intrinsic difficulty. I have indeed in an examination asked a dozen candidates, including several future Senior Wranglers, to sum the series $1 + x + x^2 \cdots$ and not received a single answer that was not

practically worthless—and this from men quite capable of solving difficult problems connected with the curvature and torsion of twisted curves.

"I cannot believe that this is due solely to the nature of the subject. There are difficulties in these ideas, no doubt; but they are not so great as many other difficulties inherent in mathematics that every young mathematician completely overcomes. The fault is not that of the subject or of the student but of the textbook and the teacher. It is not enough for the latter, if he wishes to drive sound ideas on these points well into the mind of his pupils, to be careful and exact himself. He must be prepared not merely to tell the truth but to tell it elaborately and ostentatiously. He must drill his pupils in 'infinity' and 'continuity' with an abundance of written exercises and examples as he drills them at present in poles and polars or symmetric functions or the consequences of De Moivre's theorem. Then and only then he may hope that accurate thought in connexion with these matters will become an integral part of their ordinary mathematical habit of mind."

Mathematics is a division of knowledge which is worthy of the best efforts of the ablest student. It embraces a body of facts which are interesting and definite, and it does not need either to claim what it cannot do in order to make itself popular or in elementary teaching to bring into question statements as to whose validity doubt exists. Mathematics, as Whitehead says, has nothing to do with mysteries except to explain them. The duty of the teacher is not to introduce mysteries, but to make plain the great simple facts of mathematics. Instead of an introduction into our teaching of the number infinity our great task is to get rid of it. Half of our freshmen have already learned in high school that $(1/0) = \infty$, that an equation $ax^2 + bx + c = 0$ in which $a = 0$ has the root infinity, and many other things about "infinity" that are either meaningless or false, but which tickle their fancy, excite their imagination, and make it almost impossible for them to learn mathematics. No *deus ex machina* in the form of infinite numbers can replace the hard honest work which is necessary for teacher and student in order that the latter may acquire clear and logical ideas with respect to these fundamental concepts of mathematics.

2. In dealing with imaginary numbers we are on entirely different ground from that traversed in the previous section. There is a perfectly logical method of introducing ordinary complex numbers into algebra and geometry, just as there is a logical method of introducing quaternions or numbers of any linear algebra. The methods which Professor Morley has used are particularly interesting, but the most interesting thing about them is that they give through the use of complex numbers facts about figures in a real geometry. Now that is exactly the opposite of the state of affairs that confronts us in the use of imaginary numbers in the ordinary Cartesian system.

The only questions that arise are those of interest and point of view. Why, for example, should Professor Winger criticize those who call

$$x^2 + y^2 = 0$$

a "point-circle" or single point rather than those who say that

$$x^2 + y^2 + 1 = 0$$

is an imaginary circle, when they might be more original and entertaining and say that it represents the two quaternion straight lines

$$x + yi + j = 0,$$
$$x - yi - j = 0?$$

To most people, of whom the writer is one, as long as we lack a space of four dimensions to enable us to plot all the points, real and imaginary, which satisfy an equation in two variables, it will be more interesting to confine themselves to points in the real plane, and they will not consider that they are telling only part of the truth when they say that

$$x^2 + y^2 = 0$$

represents only a single point.

II. On Teaching the Slide-Rule.

By A. A. Bennett, University of Texas.

For historical and other reasons the majority of teachers of mathematics who teach the use of logarithms are themselves unfamiliar with the use of the slide-rule. The slide-rule is rightly regarded as a necessary adjunct to an engineer's outfit of tables and instruments, but facility in handling this practical computing device is not universally acknowledged as essential to an education in pure mathematics. Most teachers who do instruct classes in the slide-rule find that the students take to it readily, and that slide-rule and logarithm table complement each other in developing a concrete notion of the value of significant figures. One of the most serious handicaps in teaching the use of the slide-rule to a large class is the difficulty of coördinating the work of teacher and student. The manufacturers of engineers' slide-rules usually have in stock large size "demonstration slide-rules" that are well fitted for the purposes of instruction. These are made of cheap material and serve also as advertisements and for these reasons may be had at a low price. These are not a new undertaking but since they are not listed in the catalogs, many instructors may have been deterred from taking up the subject in class by ignorance of the facilities available. A so-called "student's slide-rule" identical in marking with the more expensive instruments, and made of durable but cheaper material, serves all the needs of the average student in his individual study.

RECENT PUBLICATIONS.

REVIEWS.

Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen. By Heinrich W. E. Jung. Berlin and Leipzig, Walter de Gruyter & Co., 1923. 246 pages. Price $1.40.

Let there be given an irreducible polynomial in x and y of degree in y greater than zero. The equation obtained by setting such a polynomial equal to zero

will be satisfied by a function $y(x)$, termed an algebraic function of x. The domain of the independent variable x is taken as the plane of complex numbers. Let $R(x, y)$ be any rational function of x and y. If y be restricted to the values $y(x)$, $R(x, y)$ becomes a function of x, namely $R[x, y(x)]$ for which the notation $r(x)$ will be exclusively used. Fundamental in the theory of algebraic functions is the study of the nature of the zeros and poles of functions $r(x)$, and the determination of the extent to which these zeros and poles can be arbitrarily prescribed. A very important class of functions also studied in this theory is the class of the so-called abelian integrals, the integrals with respect to x of the above functions $r(x)$.

The theory of algebraic functions has been developed in several distinct ways. Riemann defined the fundamental types of abelian integrals by means of their characteristic properties, and made their existence depend upon the truth of the Dirichlet principle of the potential function theory. The fundamental existence theorems of the potential function theory necessary for a rigorous foundation of this part of Riemann's theory have only been established in recent years. With the fundamental types of abelian integrals as a basis, Riemann was then ready to develop the general theory of algebraic functions.

Clebsch and Jordan, followed by Brill and Noether, developed the theory from the point of view of algebraic geometry, first studying the intersections of algebraic curves and families of such curves, and, with this as an aid, attacking the problem of the determination of the extent to which the zeros and poles of a function $r(x)$ can be prescribed.

Weierstrass, using the powerful instruments of function theory, went directly to the problem of characterizing the functions $r(x)$. He obtained what was analogous to a partial fraction representation of such functions in terms of a simple set of such functions. He then derived the theory of abelian integrals.

While Weierstrass and others characterized the functions $r(x)$, through their principal parts at their poles, Dedekind and Weber, followed by Hensel and Landsberg, sought to determine the functions $r(x)$ by giving their zeros as well as poles. The present treatise has followed the lead of Hensel and Landsberg. Termed the "arithmetische Methode" it is essentially algebraic, with a strong tendency toward the formal. The main body of the theory is developed entirely without the aid of algebraic geometry. Use is made of Riemann surfaces throughout. The classical function theory is practically unused. For example the reviewer has found no use whatsoever made of the Cauchy formula or integral law. The basic instrument of the treatise is the power series. Such series are manipulated formally. The reader is evidently supposed competent to answer questions of convergence for himself. Although the reviewer himself prefers to develop the theory of algebraic functions from the point of view of the complex function theory, yet he found Professor Jung's treatment strikingly suggestive and unified.

The book has thirteen chapters. It starts with a chapter on the behavior of an algebraic function in the neighborhood of a point, treated with a leaning

toward strictly algebraic methods, followed by a chapter on Riemann surfaces and on the diagram of Puiseux, carefully done and illustrated by a profusion of well-selected examples. The major and distinctive portion of the book is devoted to the marshalling of theorems preparatory to the final results concerning the prescription of the zeros and poles of a function $r(x)$. First the prescribed zeros and poles are taken as finite points with nothing required as to the nature of the zeros or poles at the point at infinity. Of the functions so obtained those are sought which have prescribed properties at the point at infinity also. The existence of a desired function, say $r_1(x)$, is made to depend on the existence of a basis, that is, a finite set of functions of the type $r(x)$, upon which a fundamental requirement is that every function $r_1(x)$ be expressible as a sum of a finite number of terms, each term consisting of a member of the basis multiplied by a proper rational function of x. Matrices whose elements are functions are an essential medium in the presentation. The theory of the combination and transformation of such matrices is presented in a separate chapter. The Riemann-Roch theorem is one of the important theorems in which this part of the work culminates.

The tenth and eleventh chapters are respectively on algebraic curves in non-homogeneous coördinates, and in homogeneous coördinates. These chapters seem to the reviewer vague. There does not seem to be an exact enough connection between definitions and theorems, in particular as regards the infinite domain. The twelfth chapter is on the analysis situs of Riemann surfaces carefully written along the lines usual in connection with algebraic functions. The final chapter is on abelian integrals. The existence of the canonical integrals of the three kinds is easily established with the aid of the earlier theorems. The book ends with a proof about a page long of the fact that any abelian integral can be expressed as a sum of constant multiples of the canonical integrals plus a function $r(x)$.

The style is simple, direct, and easy to follow, although at times it seems unnecessarily formal, especially in the use of "divisors." If we except the chapters on algebraic geometry it is obvious that the writer has endeavored to give all of the proof of each of the theorems stated, and with the exception of a few places where the reader can fill in, has succeeded in doing so. A distinctive and very helpful side of the book is the numerous sets of examples worked out in a manner which splendidly supplements the main body of the theory. The book will be a valuable aid to all those who wish to take up the study of algebraic functions from the algebraic point of view.

H. M. MORSE.

Inleiding tot de Analytische Meetkunde van het platte vlak. By J. WOLFF. Groningen, P. Noordhoff, 1922. viii + 296 pages. Price f. 5.25.

Inleiding tot de Analytische Meetkunde; deel I, Het platte vlak. By J. G. RUTGERS. Groningen, P. Noordhoff, 1923. xvi + 299 pages. Price f. 7.25.

In Holland as in most countries of continental Europe, university studies are pursued almost exclusively by those who wish to prepare themselves for the

learned professions. A student who wants to take a doctor's degree in any of the natural sciences or who wants to take an engineering degree has a course in analytical geometry during his first year at the University. In the secondary school he has had thorough courses in the elementary subjects, including trigonometry, solid geometry, and sometimes mechanics, while a well-rounded general education has usually made him a person of considerable intellectual maturity.

It is for this reason that the two books under review, which present the material for this course, are a good deal more extensive, both as regards the topics included and the completeness of their treatment, and are written from a more advanced point of view than can be the case with our usual college texts in analytical geometry. While not conceived on as ample a basis as Salmon's works, they belong to the type of which the well-known text of Briot and Bouquet and the excellent book of G. Kowalewski are examples.

While these books would not therefore be suitable for our freshmen and sophomores, they are excellent companions for the more advanced student or for the exceptional undergraduate who is looking for a more complete treatment than his college course offers him. The introduction of homogeneous coördinates of points and lines, the treatment of projective properties, the discussion of general theorems on algebraic curves and the use of the theory of determinants are features in both texts. Professor Wolff devotes a final chapter to a study of projective transformations and makes general use, throughout his book, of the differential calculus; Professor Rutgers takes up some topics of the modern geometry of the triangle. More than 350 problems are scattered throughout the first of these books; there are over 200 in the second. And many of these are suitable material for students in an American college; particularly those problems which are not of the milk chocolate variety that frequently forms too large a share of the student's present diet.

A. DRESDEN.

Introduction à l'étude des Fonctions Elliptiques à l'usage des étudiants des Facultés des Sciences. By PIERRE HUMBERT. Paris, J. Hermann. 1922. 8vo. 38 pages. Price 3 francs.

Contents: Preface, p. 5; Introduction: A brief survey of some fundamental notions and results of the theory of a complex variable, pp. 7–9; Chapter I: Formation of an elliptic function, pp. 10–19; Chapter II: General remarks on elliptic functions, pp. 20–24; Chapter III: Properties of Weierstrass's functions $p(u)$, $\zeta(u)$ and $\sigma(u)$, pp. 25–37; Additional Note, pp. 37–38.

This short pamphlet may be called to the favorable attention of those who are familiar with an elementary course in complex variable theory and are not yet acquainted with the elliptic functions. Being concise and well written, it will help the reader in his study of the theory of elliptic functions as treated in the special textbooks. The existence of doubly periodic functions is shown in a very simple manner: the author considers the inverse of an elliptic integral in an analogous way to that of the study of sin u as the inverse of $u = \int_0^z (dz/\sqrt{1-z^2})$. Jacobi's functions are not to be found.

D. KAZARINOFF.

ARTICLES IN CURRENT PERIODICALS.

ACTA MATHEMATICA, volume 44, no. 1, 1923: "Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes" by G. H. Hardy and J. E. Littlewood, 1–70. [Quotation: "It was asserted by Goldbach, in a letter to Euler dated 7 June, 1742, that every even number $2m$ is the sum of two odd primes, and this proposition has generally been described as 'Goldbach's Theorem.' There is no reasonable doubt that the theorem is correct, and that the number of representations is large when m is large; but all attempts to obtain a proof have been completely unsuccessful.

"Our main result may be stated as follows: if a certain hypothesis (a natural generalization of Riemann's hypothesis concerning the zeros of his Zeta-function) is true, then every large odd number n is the sum of three odd primes."]—nos. 2 and 3, 1923: "Mémoire sur le calcul aux différences finies" by N. E. Nörlund, 71–212; "Sur l'intégrale $\int_a^x f(y)df(x)$ où x et y sont liés par une relation symétrique" by P. Appell, 213–215.

AMERICAN JOURNAL OF MATHEMATICS, volume 44, October, 1922 [published May, 1923]: "On the kernel of the Stieltjes integral corresponding to a completely continuous transformation" by C. A. Fischer, 237–246; "Equivalence and reduction of pairs of Hermitian forms" by Mayme I. Logsdon, 247–260; "Plane cubics with a given quadrangle of inflexions" by B. M. Turner, 261–278; "Normal ternary continued fraction expansions for the cube roots of integers" by P. H. Dans, 279–296; "Concerning compact Kürschák fields" by V. D. Gokhale, 297–316.

AMERICAN JOURNAL OF PSYCHOLOGY, volume 34, April, 1923: "The equation of the learning function" by M. F. Meyer and F. G. Eppright, 203–222.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, February and March, 1923: "Sur les modules des zéros des polynomes" by P. Montel, 33–34 (continuation and conclusion); "Sur les formules d'interpolation de Stirling et de Newton" by N. E. Nörlund, 35–54 (continued from volume 57); "Sur les systèmes cycliques de triples de Steiner" by S. Bays, 55–95.—volume 58, April, 1923: "Sur une classe d'équations fonctionnelles" by G. Julia, 97–128.

ANNALS OF MATHEMATICS, 2d series, volume 23, March, 1922: "Dirichlet's problem" by G. E. Raynor, 183–197; "Annihilators of modular invariants and covariants" by Olive C. Hazlett, 198–211; "Systems of linear inequalities" by W. B. Carver, 212–220; "Euler squares" by H. F. MacNeish, 221–227; "Geometric aspects of Einstein's theory" by J. Pierpont, 228–254; "Cauchy's paper of 1814 on definite integrals" by H. J. Ettlinger, 255–270; "Arithmetical deduction of Kronecker's class-number relations" by G. H. Cresse, 271–279; "Cyclotomic heptasection for the prime 43" by O. Upadhyaya, 280–281; "Summation of a double series" by T. H. Gronwall, 282–285.

ANNALS OF MATHEMATICS, 2d series, volume 23, June, 1922: "On the positions of the imaginary points of inflexion and critic centers of a real cubic" by B. M. Turner, 287–291; "Frequency distributions obtained by certain transformations of normally distributed variates" by H. L. Rietz, 292–300; "The associated point of seven points in space" by H. S. White, 301–306; "Common solutions of two simultaneous Pell equations" by A. Arwin, 307–312; "On the complete independence of Hurwitz's postulates for Abelian groups and fields" by B. A. Bernstein, 313–316; "On power series with positive real part in the unit circle" by T. H. Gronwall, 317–332; "Algebraic surfaces, their cycles and integrals. A correction" by S. Lefschetz, 333.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, March, 1923: "The twenty-ninth annual meeting of the Society" by R. G. D. Richardson, 97–116; "The Evanston meeting of the Society" by A. Dresden, 117–124; "The fifteenth regular meeting of the Southwestern Section" by E. B. Stouffer, 125–127; "A set of axioms for line geometry" by M. G. Gaba, 128–138; Reviews: by J. W. Glover of P. J. Richard and E. Petit, *Théorie mathématique des Assurances* (2 vols., 2d ed., Paris, 1922), 139–140; and by E. P. Lane and H. J. Davis of E. Cartan, *Leçons sur les Invariants Integraux* (Paris, 1922); Notes, 141–142; New publications, 143–144.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, April, 1923: "The February meeting of the Society" by R. G. D. Richardson, 145–153; "Relations between kindred Riemannian P and Q functions" by R. D. Curtis, 154–160; "The reduction of singularities of plane curves by birational transformation" [presidential address delivered before the American Mathematical Society, December 28, 1922] by G. A. Bliss, 161–183; Reviews: by R. B. McClenon of C. Tweedie, *James Stirling* (Oxford, 1922), 184–185; by J. W. Young of W. A. Granville, *The Fourth Dimension and the Bible* (Boston, 1922), 185; and by J. E. Rowe of T. Vahlen, *Ballistik* (Berlin, 1922), 186–187; Notes, 188–189; New Publications, 190–192.—May, 1923:

"The April meeting of the San Francisco Section" by B. A. Bernstein, 193–196; "The April meeting of the Society in Chicago" by A. Dresden, 197–203; "The April meeting of the Society in New York" by R. G. D. Richardson, 204–218; "An elementary proof of a fundamental lemma concerning the limit of a sum" by H. J. Ettlinger, 219–223; "Klein's collected papers, volume II" [review of Felix Klein: *Gesammelte mathematische Abhandlungen, Vol. II* (Berlin, 1922)] by V. Snyder, 224–229; "Lévy on functionals" [review of P. Lévy, *Leçons d'Analyse Fonctionnelle* (Paris,1922)] by C. A. Fischer, 229–231; Reviews: by F. Cajori of H. Andoyer, *L'Oeuvre scientifique de Laplace* (Paris, 1922), 232; by J. W. Young of F. Michel and M. Potron, *La Composition des Mathématiques dans l'Examen D'Admission à l'École Polytechnique de 1901 à 1921* (Paris, 1922), 232; by A. Emch of G. H. Ling, G. Wentworth and D. E. Smith, *Elements of Projective Geometry* (Boston, 1922), 233; by J. B. Shaw of C. E. Weatherburn, *Elementary Vector Analysis* (London, 1921), 233; by L. W. Dowling of R. Fricke, *Die elliptischen Funktionen und ihre Anwendungen, Zweiter Teil* (Leipzig, 1922), 234; and by E. P. Adams of P. Pringsheim, *Fluoreszenz und Phosphoreszenz im Lichte der neueren Atomtheorie* (Berlin, 1921), 234; Notes, 235–237; New publications, 238–240.

Bulletin des Sciences Mathématiques, second series, volume 47, April, 1923: Review by G. Valiron of E. Borel, *Méthodes et Problèmes de la Théorie des Fonctions* (Paris, 1922), 129–134; Review by H. Andoyer of Bureau des Longitudes, *Annuaire pour l'an 1923*, 135–137; Review by S. Lefschetz of Manning, *Primitive Groups. Part 1* (Stanford University, 1921), 138–139; Review by S. Lefschetz of J. B. Shaw, *Vector Calculus with Applications* (New York, 1922), 139–140; "Sur deux formules de Lagrange" by B. Niewenglowski, 141–146; "Sur l'abaissement du degré de l'équation modulaire" by J. Plemelj, 146–153; "Le passage à la limite des équations aux différences aux équations différentielles dans les problèmes aux limites" by M. Plancherel, 153–160.—May, 1923: Review by M. Brillouin of J. Villey, *Les divers aspects de la théorie de la relativité* (Paris, 1923), 161–164; Review by A. Corvisy of J. J. Thomson, *Les rayons d'électricité positive et leur application aux analyses chimiques*, 164–167; Review by R. d'Adhémar of A. E. Kennelly, *Les applications élémentaires des fonctions hyperboliques à la science de l'ingénieur électricien* (Paris, 1922), 167–170; "Le passage à la limite des équations aux différences aux équations différentielles dans les problèmes aux limites" by M. Plancherel, 170–177 (concluded); "Sur un théorème de M. Hadamard" by G. Valiron, 177–192.

Bulletin of the West Virginia Scientific Association (West Virginia University Bulletin, series 22, number 5, part 2), volume 2, number 1, April, 1923: "A slide rule representation of Einstein's restricted theory of relativity" by C. N. Reynolds, Jr., 34–41.

L'Enseignement Mathématique, volume 22, no. 6, published May, 1923: "Démonstration du théorème de Staeckel par l'élimination du temps entre les équations de Lagrange" by E. Turrière, 337–343; "Démonstration d'un théorème de Morley" by B. Niewenglowski, 344–346; "Récréation mathématique. Le Jeu de cloche et marteau" by M. Jéquier, 347–357; Mélanges et correspondance: "Fonctions triplement périodiques d'une seule variable indépendante" by M. Winants, 358–359 [Concluding sentences: "La triple périodicité d'une structure cristalline est un fait qu'aucun cristallographe ne conteste plus. Elle pourrait, et même devrait suggérer au mathématicien l'étude systématique des fonctions à trois périodes. Cette étude nous paraît fort difficile, d'autant plus que nous ne connaissons encore aucune analyse à trois dimensions, pouvant être considérée comme une généralisation de l'analyse complexe. On ne soutiendra certainement pas que la théorie des quantités complexes rentre dans celle des quaternions comme un cas particulier dans un cas général. Néanmoins, nous croyons qu'aujourd'hui l'on peut ne plus mettre en doute l'existence des fonctions triplement périodiques d'une seule variable indépendante.]; "Sur le théorème de la progression arithmétique de Dirichlet" by L. Aubry, with reply by A. M. Bedarida, 360–361; Chronique, 362–382 [contains reports of meetings of the French association for the advancement of the sciences, of the Swiss mathematical society and of the Swiss society of professors of mathematics]; Bibliographie, 382–397; Bulletin bibliographique, 398–408; Index to volume 22, 209–416.

Mathematics Teacher, volume 16, January, 1923: "The extension of concepts in mathematics" by A. J. Kempner, 1–23; "An experiment in classification of pupils in algebra" by (Miss) L. Price, 24–28; "Permutations in the 16th century Cabala" by M. Turetsky, 29–34; "The cultural value of secondary mathematics" by J. H. Minnick, 35–40; "The teaching of algebra" by H. F. Richards, 41–47; "Concerning the intercommunion of mathematics and astronomy" by Agnes G. Rowlands, 48–51; "The art of questioning" by C. G. Gould, 52–56; New books, 57–58; News notes, 59–62.—February: "Objectives in teaching of mathematics in secondary schools" by Gertrude E. Allen, 65–77; "Our geometry in Egypt and China" by W.

A. Austin, 78–86; "Mechanics" by G. R. Mirick, 87–93; "The place of the history and recreations of mathematics in teaching algebra and geometry" by L. G. Simons, 94–101; "A study of the cultivation of space imagery in solid geometry through the use of models" by E. W. Schreiber, 102–111; "Some aspects of correlation theory" by L. E. Mensenkamp, 112–122; "Concerning the disciplinary value of mathematics" by C. J. Keyser, 123; News and notes, 124–128.—March: "The Thorndike philosophy of teaching the processes and principles of arithmetic" by M. A. Bailey, 129–140; "Probability applied to grades" by E. J. Moulton, 141–149; "The cultural value of mathematics" by Helen E. Howarth, 150–156; "Defects remaining in the notation and nomenclature of elementary mathematics" by J. V. Collins, 157–161; "The logic of mathematical processes" by H. F. Sloctemyer, 162–169; "The way mathematicians work" by J. B. Shaw, 170–174; "Informal tests for diagnosis and remedial teaching in mathematics" by P. L. Spencer, 175–182; "Recent symbolisms for decimal fractions" by F. Cajori, 183–187; News and notes, 188–190; Discussion, 191–192.

MATHEMATICS TEACHER, volume 16, April, 1923: "Psychological tests of mathematical ability and educational guidance" by Agnes L. Rogers, 193–205; "Textbooks in unified mathematics for college freshmen" by Vera Sanford, 206–214 [Quotation from conclusion: "If one may make a guess, the future development of this work will be less and less along the conventional line. As one studies these textbooks, he almost invariably finds that the work in analytics is arid in comparison to the rest. One questions whether the normal probability curve, though younger, may not in time usurp some of the attention paid to conics. . . . The more radical of the books succeed in convincing us that freshman college mathematics may be made to present many contacts with the real world, and that, even though these contacts may be through problems that have only the appearance of reality, they at least have the virtue of showing the things that 'mathematics has done and is doing for mankind.' "]; "Teaching the algebraic language to Junior High School pupils" by J. R. Overman, 215–217; "The unitary organization of the mathematics of the seventh, eighth and ninth grades" by E. R. Breslich, 228–235; "Mechanics" by G. R. Mirick, 236–241; "A brief study in non-mathematical logic" by N. J. Lennes, 242–246; "Classification of positive integers as regards the ultimate sum of their digits" by G. A. Miller, 247–248; "Team work in elementary algebra" by J. B. Hawley, 248–250; News and notes, 251–256.—May, 1923: "Empirical theorems in Diophantine analysis" by R. D. Carmichael, 257–265; "The Pennsylvania state course of study in mathematics" by J. A. Foberg, 266–273; "Fate and Freedom" by A. Korzybski, 274–290; "Mechanics" by G. A. Mirick, 291–294; "Varieties of minus signs" by F. Cajori, 295–301; "Correlation of the mathematical subjects develops mathematical power" by C. A. Stone, 302–310; News and notes, 311–314; New books, 315–319.

MATHEMATISCHE ANNALEN, volume 88, nos. 3–4 (published February 17, 1923): "Bemerkungen über unendliche Folgen und ganze Funktionen" by G. Pólya, 169–183; "Additive Theorie der Zahlkörper II" by C. L. Siegel, 184–210; "Ueber ein Problem von A. Hurwitz, quaternäre quadratische Formen betreffend" by H. Brandt, 211–214; "Ueber S. Lies Geometrie der Kreise und Kugeln" (continuation) by E. Study, 215–241; "Ueber die invariante Darstellung algebraischer Funktionen einer Veränderlichen" by K. Petri, 242–289; "Beiträge zur allgemeinen Topologie I" by H. Tietze, 290–312.

MATHEMATISCHE ANNALEN, volume 89, nos. 1–2, published April 24, 1923: "Die Flächen vierter Ordnung mit gewundener Doppelkurve" by H. Mohrmann, 1–31; "Ueber Kurven von Maximal-Klassenindex. Ueber Kurven von Maximalindex" by J. Sz. Nagy, 32–75; "Ueber Axiomensysteme für beliebige Satzsysteme, Teil II. Sätze Höheren Grades" by P. Herz, 76–102; "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I" by K. Löwner, 103–121; "Ueber die Gewinnung summierbarer Polynomreihen aus summierbaren Fourierreihen" by K. Popoff, 122–125; "Bemerkung zu der Arbeit des Herrn Popoff 'Ueber die Gewinnung summierbarer Potenzreihen aus summierbaren Fourierreihen' " by O. Blumenthal, 126–129; "Oszillationstheoreme oberhalb der Stieltjesschen Grenze" by O. Haupt and E. Hilb, 130–146; "Ueber die Zetafunktionen gewisser algebraischer Zahlkörper" by E. Artin, 147–156; "On a formula of transformation" by P. O. Upadhyaya, 157–159.

MATHEMATISCHE ZEITSCHRIFT, volume 13, nos. 3–4: "Axiomatische Begründung der transfiniten Kardinalzahlen, I. (Herrn K. Hensel zum sechzigsten Geburtstag.)" by A. Fraenkel, 153–188; "Kurven auf algebraischen Flächen" by H. W. E. Jung, 189–201; "Ebene Schnitte und Berührungskegel einer algebraischen Fläche" by H. W. E. Jung, 202–216; "Die Tangentenlosigkeit der von Kochschen Kurve" by F. Apt, 217–222; "Die windschief involutorischen Paarungen in einer linearen Strahlenkongruenz und die beiden Arten windschiefer involutorischer linearer Strahlenkongruenzen" by S. Jolles, 221–262; "Zur Theorie der nichtlinearen Differential-

gleichungen" by J. Horn, 263–282; "Ueber die Riemannsche Funktionalgleichung der ζ-Funktion. (Dritte Mitteilung.) Die Funktionalgleichung der L-Reihen" by H. Hamburger, 281–311; Mathematische Preisaufgabe der Fürstlich Jablonowskischen Gesellschaft: (Quoted) "Der Gausssche Algorithmus des 'arithmetisch-geometrischen Mittels' kann dahin verallgemeinert werden, dass an Stelle der beiden Mittelbildungen irgend zwei Funktionen φ und ψ von zwei Veränderlichen treten. Man kommt auf diese Weise, ausgehend von zwei beliebigen Zahlwerten x und y, auf zwei unendliche Folgen

$$\begin{array}{llllll} x, & x_1, & x_2, & x_3, & \cdots, \\ y, & y_1, & y_2, & y_3, & \cdots, \end{array}$$

welche durch die rekurrenten Gleichungen

$$\begin{array}{l} x_n = \varphi(x_{n-1}, y_{n-1}) \\ y_n = \psi(x_{n-1}, y_{n-1}) \end{array} \qquad (n = 1, 2, 3, \cdots),$$

beherrscht sind. Im allgemeinen werden dann keine Grenzwerte jener Folgen existieren, während doch die Folgen selbst besondere Eigenschaften besitzen können.

"Die Gesellschaft wünscht:

"Die Untersuchung der genannten Folgen für irgendwelche einfache Funktionen $\varphi(x, y)$, und $\psi(x, y)$, wobei namentlich auch eine independente Darstellung der rekurrent definierten Glieder der Reihen ins Auge zu fassen wäre.

"Einlieferung bis zum 31. Oktober 1924; Preis 3000 Mark," 312.

MATHEMATISCHE ZEITSCHRIFT, volume 16, April, 1923: "Untersuchungen über die Konvergenz der limitärperiodischen Jacobi-Ketten beliebiger Ordnung" by E. Schlechter, 173–206; "Sur le théorème d'unicité des solutions des équations différentielles ordinaires" by J. Tamarkine, 207–213; "Sur la méthode de C. Störmer pour l'intégration approchée des équations différentielles ordinaires" by J. Tamarkine, 214–219; "Momentprobleme für ein endliches Intervall" by F. Hausdorff, 220–248; "Beiträge zur Theorie der Kurven konstanter geodätischer Krümmung auf krummen Flächen" by R. Weyrich, 249–272; "Ueber die Gestalt der Integralkurven einer Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes" by O. Perron, 273–295; "Ueber eine Integralgleichung in der Theorie der heterogenen Gleichgewichtsfiguren" by J. Lense, 296–300; "Ueber die Lösungen der Riemannschen Funktionalgleichung" by E. Hecke, 301–307; "Ueber die Parallelverschiebung in Riemannschen Räumen" by H. Tietze, 308–317; "Ganzzahlige Gleichungen ohne Affekt" by M. Bauer, 318–319.

MATHEMATISCHE ZEITSCHRIFT, volume 17, May, 1923: "Verallgemeinerung eines Minkowskischen Satzes" by R. Remak, 1–34; "Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen" by H. Prüfer, 35–61; "Untersuchungen über die Figur der Himmelskörper. Vierte Abhandlung. Zur Maxwellschen Theorie der Saturnringe" by L. Lichtenstein, 62–110; "Ueber die Bianchische Identität für symmetrische Uebertragungen" by J. A. Schouten, 111–115; "Ueber die Maximalzahl der Doppelpunkte bei ebenen Polygonen von gerader Seitenzahl" by E. Steinitz, 116–129; "Ueber das Wachstum analytischer Funktionen in Halbstreifen und ähnlichen Gebieten" by L. Neder, 130–143; "Ueber affine Geometrie: Zur Theorie der Affingesimsflächen" by E. Salkowski, 144–148; "Ueber einen Grenzwertsatz" by O. Perron, 149–152; "Bilineare Transformation quadratischer Formen" by H. Brandt, 153–160.

MESSENGER OF MATHEMATICS, volume 52, October, 1922: "The transformation of the elliptic function of the seventh order" by G. Greenhill, 81–89; "A proof of Burnside's formula for $\log \Gamma(x + 1)$ and certain allied properties of Riemann's ζ-function" by J. R. Wilton, 90–93; "A direct proof of the binomial theorem for a rational index" by T. W. Chaundy, 94–96.—November, 1922: "A direct proof of the binomial theorem for a rational index" (continued) by T. W. Chaundy, 97–98; "Adaptation of curvilinear isothermal coördinates to integrate the equations of equilibrium of elastic plates" by B. G. Galerkin, 99–109; "Sur quelques séries et produits infinis" by S. P. Sorensen, 109–112.—No. 8, December, 1922: "The connexion between the sum of the squares of the divisors and the number of the partitions of a given number" by P. A. MacMahon, 113–116; "Electromagnetism and dynamics" by H. Bateman, 116–128.

NATURE, volume 110, November 25, 1922: "The time-triangle and time-triad in special relativity" by R. A. P. Rogers, 698–699; "Space-time geodesies" by H. T. H. Piaggio, 699.—December 16: "Space-time geodesics" by A. A. Robb, 809–810.—December 23: "A quantum theory of optical dispersion" by C. G. Darwin, 841–842.—Volume 111, January 6, 1923: Notices by W. E. H. B. of A. MacLeod, *Introduction à la géométrie non-Euclidienne* (Paris, 1922) and of L. E. Dickson, *History of the Theory of Numbers* (Vol. II, Washington, 1920).— January 20:

"Early mathematical instruments in Oxford" [review of R. T. Gunther's *Early Science in Oxford* (London, 1922)], 75–76.—February 3: "Greek geometry with special reference to infinitesimals" by T. L. Heath, 152–153; "The finitistic theory of space" [notice of N. M. Poppovich's *Die Lehre vom diskreten Raum in der neueren Philosophie* (Vienna and Leipzig, 1922). "The theory leads Dr. Petronievics to affirm the absoluteness of Euclidean space"].—February 10: "The definition of limiting equality" by G. H. Bryan, 183–184 [Quotation: "I consider that the proper test of limiting equality is that the difference between two quantities should become (numerically) less than any assignable fraction of one of the quantities, however small, in other words that $x - a < ae$ where e is any fraction of unity, however small (instead of $x - a < e$ where e is any quantity, however small), the present definition being assumed to hold good even if the two quantities vanish or become infinite at the limit"]—February 17: "Stirling's theorem" by J. Satterly, 220; "Definitions and laws of motion in the 'Principia'" by G. Greenhill, 224–226.—February 24: "Absolute measure and the C. G. S. units" by G. Greenhill, 259–261; "The teaching of elementary geometry," 271.—March 3: "Sequence in school geometry," 277–278.—March 17: Notice of J. L. Heiberg, *Mathematics and Physical Science in Classical Antiquity* (trans. D. C. Macgregor, London, 1922), 355–356; Obituary notice of A. N. Favaro, by F. Cajori, 368.—March 24: Notice by E. G. C. Poole of J. Edwards, *A Treatise on the Integral Calculus* (Vol. 2, London, 1922), 391; "Definitions and laws of motion in the 'Principia'" by W. Peddie, 395, and by F. E. Hackett, 395–396; "Stirling's theorem" by J. Strachan, 397; Obituary notice of James Gow, by T. L. Heath, 403.—March 31: "The Fourier-Bessel function" [review of A. Gray and G. B. Mathews, *A Treatise on Bessel Functions and their Applications to Physics* (2d ed., prefaced by A. Gray and T. M. Macrobert, London, 1922), and of G. M. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge, 1922)] by G. Greenhill, 422–425.—April 28: Notice of E. Study, *Mathematik und Physik* (Braunschweig, 1923), 565; of H. P. Few, *Elementary Determinants for Electrical Engineers* (London and New York, 1922), 566.

NATURE, volume 111, May 5, 1923: "Stirling's theorem" by H. E. Soper, 601.—May 12: "Martin's equations for the epidemiology of immunizing diseases" by A. J. Lotka, 633–634.—May 26: "New works on relativity" [review of ten recent books, including L. Silberstein, *The Theory of General Relativity and Gravitation* (Toronto, 1922), and F. D. Murnaghan, *Vector Analysis and the Theory of Relativity* (Baltimore, 1922)], 697–699.—June 2: "An Einstein paradox" by R. W. Genese, 742.—June 9, 1923: "Actuarial mathematics" by W. E. H. B. [review of A. Henry, *Calculus and Probability for Actuarial Students* (London, 1922), and of E. F. Spurgeon, *Life Contingencies* (London, 1922)], 769–770; "A puzzle paper band" by C. V. Boys, 774.—June 16: "Martini's equations for the epidemiology of immunizing diseases" by G. N. Watson, 808; "The tercentenary of Blaise Pascal" by H. W. Carr, 814–816; "Curve fitting," 824.—June 23: "The teaching of the calculus" by G. H. B. [review of G. W. Brewster, *Common Sense of the Calculus* (London, 1923)], 837–838.

NOUVELLES ANNALES DE MATHÉMATIQUES, fifth series, volume 1, May, 1923: "Sur les systèmes de quadriques ayant mèmes projections de leurs lignes de courbure sur un plan principal commun" by M. d'Ocagne, 277–284; "Sur la formule d'Holditch et les applications qu'on peut en déduire" by R. Esteve, 284–300; "Sur un invariant intégral se rattachant à la Mécanique statistique" by J. Haag, 301–302.—June, 1923: "Sur les fonctions analytiques d'une variable réelle" by G. Valiron, 321–329; "Sur une forme remarquable de l'intégrale de l'équation des cordes vibrantes" by S. Zaremba, 330–338; "La construction du centre de courbure des coniques d'après Mannheim démontrée par le théorème de Pascal" by J. Larras, 338–339; "Sur les cercles focaux" by H. Lebesgue, 340–350; "Sur un probleme de choc avec frottement" by H. Villat, 351–357.

PEDAGOGICAL SEMINARY, volume 30, March, 1923: "Mathematics in current literature" by Martha MacLear, 48–50; "Self-taught arithmetic from the age of five to the age of eight" by Sophie R. Altshiller-Court, 51–68.

POPULAR ASTRONOMY, volume 31, May, 1923: "Leap year rules and calendar accuracy" by C. F. Marvin, 298–308. [Quotation: "The present rule (Gregorian calendar) had best be continued to the year 3200 A.D., when the error of reckoning will be about 0.9 day. Then introduce a rule to omit 5 leap years in 600 years, resulting in a calendar which will operate until the year 17600 with errors well under one day throughout the whole period."]

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY, volume 40, 1922: "Determination of the arbitrary constants which appear in the asymptotic expansions for the functions of the elliptic cylinder" by W. Marshall, 2–8; "Newton's conception of a limit as interpreted by Jurin and Robins respectively" by G. A. Gibson, 9–20; "Notes on Everett's interpolation

formula" by G. J. Lidstone, 21–26; "On Mathieu functions of higher order" by P. Humbert, 27–33; "The vibrations of a particle about a position of equilibrium—Part 2. The relation between the elliptic function and series solutions" by B. B. Baker, 34–49; "A dissection of the extended form of Pythagoras' Theorem" by G. D. C. Stokes, 50–54; "A new form for the sum of a trigonometric series" by G. J. Lidstone, 54–56; "To construct an ellipse whose focus F is given, which shall pass through three given points A, A', A'' (Halley's Problem)" by R. F. Davis, 56–57; "Common Logarithms calculated by simple multiplication" by R. F. Muirhead, 57–60; "Trigonometrical Ratios of $(A \pm B)$" by A. D. Russell, 60–61; "On Sylvester's Dialytic Method of Elimination" by E. T. Whittaker, 62–63.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. A., vol. 8, December, 1922: "Projective and affine geometry of paths" by O. Veblen, 347–350; "A further note on the mathematical theory of population growth" by R. Pearl and L. J. Reed, 365–368.—Volume 9, January, 1923: "On Riemann spaces conformal to Euclidean space" by H. W. Brinkmann, 1–3; "Equiaffine geometry of paths" by O. Veblen, 3–4; "Affine geometries of paths possessing an invariant integral" by L. P. Eisenhart, 4–7; "Closed connected sets which are disconnected by the removal of a finite number of points" by J. R. Kline, 7–12; "Some extensions in the mathematics of hydromechanics" by R. S. Woodward, 13–18.—February: "Sets of conjugate cycles of a substitution group" by G. A. Miller, 52–54.—March: "Note on the relative abundance of the elements in the earth's crust" by A. J. Lotka, 87–90; "Continuous transformations of manifolds" by S. Lefschetz, 90–93; "A lemma on systems of knotted curves" by J. W. Alexander, 93–95.—April: "Concerning the cut-points of continuous curves and of other closed and connected point-sets" by R. L. Moore, 101–106; "On Riemann spaces conformal to Einstein spaces" by H. W. Brinkmann, 172–174.—Vol. 9, June, 1923: "Another interpretation of the fundamental gauge-vector of Weyl's theory of relativity" by L. P. Eisenhart, 175–178; "Tensor analysis without coördinates" by G. Y. Rainich, 179–183; "Geometric aspects of the Abelian modular functions of genus four (III)" by A. B. Coble, 183–187.

SCHOOL AND SOCIETY, volume 17, January 29, 1923: "A mathematician on the present status of the formal discipline controversy" by N. J. Lennes, 63–71 [presented at the Rochester meeting of the Association, September, 1922].—March 31: "Mathematics the project instrument" by T. Lindquist, 343–348.—May 19, 1923: "The Ph.D. degree and honesty" by G. A. Miller, 553.

SCHOOL REVIEW, volume 31, May, 1923: "Diagnostic algebra tests and remedial measures" by T. M. Deam, 376–379.

SCIENCE, new series, volume 57, April 20, 1923: "The Pi Mu Epsilon mathematical fraternity" by E. D. Roe, Jr., 461–462.—May 4: "An Egyptian mathematical papyrus in Moscow" by L. C. Karpinski, 528–529; "The international work of scientific synthesis" [a review of the international journal *Scientia*] by R. D. Carmichael, 529–532.—May 11: "Note on the equations for tortuosity in Thomson and Tait's *Natural Philosophy*" by C. Barus, 562.—May 25: "The American Mathematical Society" [report of the meeting of the Chicago Section, April 13–14, 1923] by A. Dresden, 620.

SCIENCE, new series, volume 57, June 1, 1923: Review by R. D. Carmichael of A. Einstein, *The Meaning of Relativity* (Princeton lectures translated into English by E. P. Adams, Princeton, 1922), 642–643. [Quotation: "Full details cannot be given in so short an exposition. But the author can and does succeed in making clear the general ideas and in setting before the reader the spirit and trend of the argument and in leading him to see much of the detail of the whole theory. The exposition is thus a very useful one."]—June 8: "Mathematical propaganda" by G. A. Miller, 663.—June 22: "The new volumes of the Encyclopedia Britannica" by R. D. Carmichael, 721–724.

TÔHOKU MATHEMATICAL JOURNAL, volume 21, July, 1922: "A new proof of a theorem of Taylor's" by Y. Tanaka, 1–2; "On the theory of double sequences" by T. Kojima, 3–14; "On the curvature of the closed convex curve" by T. Kojima, 15–20; "Notes on closed convex curves" by T. Kubota, 21–25; "On the Fourier constants and a special system of non-homogeneous linear equations in infinitely many unknowns" by S. Takenaka, 26–42; "Note on the rapidly convergent series for $N!$ due to Prof. Burnside" by S. Nakajima, 43–45; "Cyclotomic sexe-section for the primes 19 and 31" by P. O. Upadhyaya, 46–50; "Über gewisse Infinitesimaloperationen der höheren Operationsstufen" by R. E. Moritz, 51–64; "Pentaspherical geometries in noneuclidean space, II" by T. Takasu, 65–137; "On the multiplication table in China" by K. Ueno, 138–147; "The logarithm in Japanese mathematics" by T. Hayashi, 148–190.—Volume 22, December, 1922: "On a Diophantine equation" by K. Oishi, 1–13; "Sur le champ de gravitation dans l'espace vide" by K. Ogura, 14–37; "Untersuchungen über das Poissonsche

Integral auf der Kugel und seine Ableitungen" by P. E. Strässle, 38–76; "An extension of a theorem of Salmon" by Y. Sawayama, 77–78; "Über eine arithmetische Eigenschaft gewisser Reihenentwicklungen" by G. Pólya, 79–81; "Sur certains systèmes d'équations différentielles" by P. Sibirani, 82–86; "Über die Mittelwerte analytischer Funktionen" by G. Szegö, 87–98; "Axiomatic investigation on number-systems, I" by K. Yoneyama, 99–137; "On multiplicative and enumerative properties of numerical functions" by E. D. Pepper, 138–152; "On geometrical construction by a ruler of finite length and compasses of finite aperture" by M. Shibayama, 153–154; "On a sextic" by P. O. Upadhyaya, 155–157; "Cyclotomic sexe-section for the prime 61" by P. O. Upadhyaya, 158–162; "Second paper on tautochronous motion" by P. O. Upadhyaya, 163–164; "On the integral $\int_0^\infty \frac{\sin^n x}{x^m}\,dx$ with an appendix on its application to theory of approximation of a function" by T. Hayashi, 165–170.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 22, October, 1921 [published April, 1923]: "On certain numerical invariants of algebraic varieties with application to Abelian varieties" by S. Lefschetz [concluded], 407–482; "On restricted systems of higher indeterminate equations" by E. T. Bell, 483–488; "Maximum modulus of some expressions of limited analytic functions" by S. Kakeya, 489–504; "Differential variations in ballistics, with applications to the qualitative properties of the trajectory" by T. H. Gronwall, 505–525; "An expansion theorem for a system of linear differential equations of the first order" by W. A. Hurwitz, 526–543.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 54, no. 2, published May 5, 1923: "Vom Ptolemäischen zum Kopernikanischen System" by W. Lietzmann, 65–70; "Zur Bestimmung der Asymptoten einer ebenen Kurve" by V. Kommerell, 70–75; "Zu den arithmetischen Operationen" by N. Gennematás, 75–78; "Ueber die Behandlung der Parallelenlehre im Unterricht" by K. Fladt, 78–84; "Beiträge zur Berechnung des Kugeldreiecks im Falle des casus ambiguus" by Ruoss, 84–89; "Ueber den Zusammenhang der Keplerschen Gesetze untereinander" by H. Teege, 89–94; "Ein elementares Verfahren, die relativistischen Aberrationsgesetze unmittelbar aus dem Diagramm der Aberration abzulesen" by H. Meurer, 95–96; "Die Winkelmessung der Artilleristen" by P. Luckey, 97–99; "Sphärisch-astronomische Relation zwischen den Neigungswinkeln der regulären Körper" by H. Michnik, 99; Aufgabenrepertorium, 100–105; Berichte, 106–110; "Aus der Forschung. Eine amerikanische Geschichte der Zahlentheorie" [review of L. E. Dickson, *History of the Theory of Numbers* (Vols. I and II, Washington, 1919–1920)] by H. Wieleitner, 110–111; Bücherbesprechungen und Zeitschriftenschau, 113–127; "Die Lehre von den Wurzeln im elementaren Unterricht" by Brettar, 127–128.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Tex.

CLUB ACTIVITIES.

THE JUNIOR MATHEMATICS CLUB OF THE UNIVERSITY OF CHICAGO, Chicago, Ill.
[1921, 324.]

Officers elected for the year 1922–1923 were: President (fall and winter), C. E. Van Horn, Gr.; president (spring), L. M. Graves, Gr.; secretary and treasurer, Margaret Mauch, Gr.; chairman of program committee, Mrs. J. P. Ballantine, Gr.; chairman of social committee, Esther Weaver, Gr. Two social meetings were held during the year. At the regular meetings the following papers were presented:

October 25, 1922: "Difference quotients" by J. P. Ballantine, Gr.
November 8: "Mathematics, mathematicians and humor" by M. H. Ingraham, Gr.
December 6: "Non-euclidean geometry" by L. M. Graves, Gr.
January 10, 1923: "Some simple methods of deriving series" by Mrs. J. P. Ballantine, Gr.
January 24: "Classes, sets and some paradoxes" by C. E. Van Horn, Gr.
February 7: "The fourth dimension" by D. S. Patton, Gr.
February 21: "Coefficients of correlation" by H. R. Phalen, Gr.
April 11: "The game of Nim" by Esther Weaver, Gr.
April 25: "Caustic curves" by D. L. Holl, Gr.

(Reported by Miss Mauch.)

THE MATHEMATICS CLUB OF HAMLINE UNIVERSITY, St. Paul, Minn.

This club was organized in the spring of 1923, Professor Roger A. Johnson acting as president, and Victor Westman '24 as secretary. The following papers were presented:
March 28, 1923: "Nim and other mathematical recreations" by Mr. C. A. Rupp, instructor.
April 25: "The theory of relativity" by Stanley Poukey '23.
May 23: "Some aspects of non-euclidean geometry" by Professor Johnson.

(Reported by Professor Johnson.)

THE MATHEMATICS CLUB OF THE STATE UNIVERSITY OF IOWA, Iowa City, Ia.

[1922, 355.]

As officers for the year 1922–1923 the following were elected: President, Orley Brown, graduate assistant; secretary, Glenn Aldrich, graduate assistant. The meetings, all open to the public, were well attended. Papers were read as follows:
October 12, 1922: "Purposes of the Mathematics Club" by Professor W. H. Wilson.
November 2: "Magic squares" by Helen Moon, Gr.
November 16: "Some phases of projective geometry" by Orley Brown, Gr.
December 7: "Mathematical recreations" by Iona Reger '23.
January 18, 1923: "Mathematical tricks and puzzles" by Glenn Aldrich, Gr.
February 1: "The development of algebraic notation" by Frances Baker '23.
February 15: "The cubic equation" by Eric Erickson, Gr.
March 1: "The theory of probability" by Henry Pollard, Gr.
March 15: "The ellipse" by Ruth Balluf '23.
April 12: "Various kinds of averages" by Clarence Balof, Gr.
May 3: "Logarithms of complex numbers" by Howard Hughes '23. Election of officers: President, Eric Erickson, Gr.; secretary, Howard Hughes '23.

(Reported by Mr. Aldrich.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3040. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Give the radius, R, of a sphere rolling down two intersecting straight lines including the angle 2α and equally inclined to the horizon; show that the locus of the center of the sphere is an ellipse of semi-axes $R\csc\alpha$, R.

3041. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line passing through a fixed point.

3042. Proposed by C. N. SCHMALL, New York City.

Given the bases and the sum of the areas of several triangles that have a common vertex; show that the locus of the vertex is a straight line.

3043. Proposed by O. D. KELLOGG, Harvard University.

Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)dxdy = 0 \tag{1}$$

for all functions V with continuous second derivatives and vanishing on the boundary of T, U is harmonic, *i.e.*, satisfies Laplace's equation $(\partial^2 U/\partial x^2)+(\partial^2 U/\partial y^2)=0$. (It is understood that if U becomes infinite in the neighborhood of a boundary point of T, V is so further restricted that the integral shall have a sense.)

SOLUTIONS.

2981 [1922, 313]. Proposed by H. P. MANNING, Providence, R. I.

Find the envelope of the circle on which two diametrically opposite points divide in a given ratio the focal radii of a variable point on an ellipse or hyperbola.

Solution by William Hoover, Columbus, Ohio.

Let C be a point on a central conic with foci A and B, and center O. If Q and Q' are points on BC and AC such that $BQ/BC = AQ'/AC = k$, then the diameter, QQ', of the circle of the problem is equal to $(1-k)AB = 2(1-k)ae$, CO passes through the center C' of the circle, and $OC' = kOC$. Thus C' describes a conic similar to the original, and the envelope of the equal circles must consist of two branches parallel to the conic C'. If the coördinates of C, in case of the ellipse, are $a\cos\varphi$, $b\sin\varphi$, then the coördinates of C' are $ka\cos\varphi$, $kb\sin\varphi$. Let ψ be the inclination of the normal at C', then $\tan\psi = a\sin\varphi/b\cos\varphi$, $\sin\psi = \sin\varphi/\sqrt{1-e^2\cos^2\varphi}$. Hence

$$x = ka\cos\varphi \pm \frac{(1-k)be\cos\varphi}{\sqrt{1-e^2\cos^2\varphi}},$$
$$y = kb\sin\varphi \pm \frac{(1-k)ae\sin\varphi}{\sqrt{1-e^2\cos^2\varphi}}.$$

The hyperbola may be treated in a similar manner.

Also solved by A. Pelletier, J. B. Reynolds and J. K. Whittemore.

2988 [1922, 356]. Proposed by PHILIP FRANKLIN, Harvard University.

Prove, geometrically, that if in an ellipse the tangent at P cuts the directrices in Z, Z' and the remaining tangents from Z and Z' to the ellipse meet at T, PT is the normal to the ellipse at P. (An analytic proof is given in the *Journal of the Indian Mathematical Society*, vol. 13, 1921, p. 234.)

Solution by Nathan Altshiller-Court, University of Oklahoma.

The line QQ' joining the points of contact Q, Q' of the tangents TZ, TZ' with the given ellipse (E) is the polar of the point T, hence PT is the polar of the point $R \equiv (QQ', PZZ')$; therefore, the four points Q, Q', R, $S \equiv (QQ', PT)$ are harmonic, and so is the pencil $P(QQ', RS)$. Now the lines PQ, PQ' are the polars of Z, Z' with respect to (E), and since Z, Z' lie on the directrices of (E), the lines PQ, PQ' pass through the foci of (E). Thus the line PT is separated harmonically from the tangent PZZ' by the focal radii PQ, PQ' of P, *i.e.*, PT is the normal to (E) at P. For $\angle QPZ = \angle Q'PZ$, a well-known property, and hence $\angle QPS = \angle Q'PS$.

Also solved by J. W. Clawson, R. M. Mathews, A. Pelletier, Mabel M. Young and the Proposer.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Because the Mathematical Association meets at the University of Cincinnati in December in affiliation with the American Association for the Advancement of Science, railroad rates are available for a fare and a half. The reduced rates should prove attractive to the Association members and guarantee a large attendance from the Middle West, as well as from points farther west and east. President Carmichael's presidential retiring address and other important papers will be given at the first session on Thursday afternoon, December 27. A second session will be held on Friday morning and a joint session on Friday afternoon, in conjunction with Section A of the American Association and the Chicago Section of the American Mathematical Society, at which time addresses in commemoration of the seventy-fifth anniversary of the American Association will be given by Professors G. A. Miller and A. B. Coble, and, at the invitation of the Chicago Section and this Association, Professor L. E. Dickson will speak on "Arithmetics and their Algebras."

The following 19 doctorates with mathematics as major subject were conferred by American Universities in the academic year 1921–1922; the university and the title of the dissertation are given with each name. C. R. ADAMS, Harvard, "The general theory of linear partial q-difference equations and of the linear partial difference equation of the intermediate type"; B. H. BROWN, Harvard, "The equilong transformations of Euclidean space"; MARGARET BUCHANAN, Bryn Mawr, "Systems of two linear integral equations with two parameters and symmetrisable kernels"; E. H. CARUS, Chicago, "Invariants as products: a vector interpretation of the symbolic method"; W. E. CEDERBERG, Wisconsin, "On the solution of the differential equations of motion of a double pendulum"; H. S. EVERETT, Chicago, "Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials"; V. D. GOKHALE, Chicago, "Concerning compact Kürschák Fields"; CLARIBEL KENDALL, Chicago, "Certain congruences determined by a given surface"; R. E. LANGER, Harvard, "I. Developments associated with a boundary problem not linear in the parameter. II. The boundary problems and developments associated with a system of linear differential equations of the first order"; C. C. MACDUFFEE, Chicago, "Invariantive characteristics of linear algebras with the associative law not assumed"; E. D. MEACHAM, Chicago, "Properties of surfaces whose osculating ruled surfaces belong to linear complexes"; EUGENIE M. MORENUS, Columbia, "Geometric properties completely characterizing the set of all the curves of constant pressure in a field of force"; ANNA M. MULLIKIN, Pennsylvania, "Certain theorems relating to plane connected point sets"; ELEANOR PAIRMAN, Radcliffe, "Expansion theorems for solution of a Fredholm's linear homogeneous integral equation of the second kind with kernel of special

non-symmetric type"; H. P. PETTIT, Illinois, "A general cyclide with special reference to the quintic cyclide"; J. F. REILLY, Iowa, "On certain generalizations of osculatory interpolation"; W. P. UDINSKI, Illinois, "On a series of rational functions formally analogous to Fourier's series"; B. C. WONG, California, "A study and classification of ruled quartic surfaces by means of a point-to-line transformation"; E. B. ZEISLER, Chicago, "Definite integral representation of invariants."

The following five doctorates, conferred in the latter half of the calendar year 1922, are listed in the same way as the preceding. E. H. CLARKE, Chicago, "On the minimum of the sum of a definite integral and a function of a point"; J. D. ESHLEMAN, Chicago, "The Lagrange problem in parametric form in the calculus of variations"; W. S. KIMBALL, Chicago, "Scattering of particles by an Einstein curve"; HARRY LANGMAN, Columbia, "Conformal transformations of period n and groups generated by them"; J. S. TURNER, Chicago, "Fundamental system of formal invariants of a modular group of transformations."

The following four doctorates in mathematical physics or mathematical astronomy were conferred during the calendar year 1922; the university, the major subject, and the title of the dissertation are given with each name. F. E. CARR, Chicago, mathematical astronomy, "A solution of the problem of two bodies one of which is a rotating oblate spheroid"; H. C. LEVINSON, Chicago, mathematical astronomy, "The gravitational field of masses relatively at rest according to Einstein's theory of gravitation"; LOUIS SLICHTER, Wisconsin, mathematical physics, "An experimental study of an acoustic system"; WARREN WEAVER, Wisconsin, mathematical physics, "A summary of the analytic formulation of the theory of electrodynamics."

The many friends of SAMUEL DOUGLAS KILLAM, professor of mathematics at the University of Alberta, will be shocked to learn of his death by drowning at Lake Annis, Nova Scotia, on July 22, 1923. He was born in Yarmouth, Nova Scotia, May 9, 1888, and received the degrees of B.A. and M.A. from Mount Allison University in 1908 and 1910 respectively. He spent the winter semester of 1908–1909 at the University of Berlin but then went to the University of Göttingen where he received the Ph.D. degree in 1912. His thesis (28 pp. and 15 plates) was entitled *Ueber graphische Integration von Funktionen einer komplexen Variabeln mit speziellen Anwendungen.* In 1912 he was appointed instructor in mathematics at the University of Rochester; in the following year he went as instructor in mathematics to the University of Alberta where he remained ever since except for a period of war service in the Canadian Artillery. In 1915 he was promoted to be assistant professor of applied mathematics and he was promoted to a full professorship in 1921. In May of last year he presented to the Royal Society of Canada a paper on "The solution of plane triangles by nomographic charts." He published an article on "Bond schedules for the amortization of a premium or accumulation of a discount" in the *Canadian Chartered Accountant,* April, 1920. He has been a member of the Association since 1920.

JOHN GASTON LEATHEM died March 19, 1923. He was born in Belfast, Ireland, on the 5th of May, 1871. After studying at Queen's College, Belfast, he went to Cambridge in 1891 and was fourth wrangler in the tripos of 1894. He held the Isaac Newton studentship in astronomy and physical optics from 1896–99 and was mathematical lecturer at St. John's College, Cambridge, afterwards becoming university lecturer. In 1905 he took up, in conjunction with E. T. Whittaker, the editorship of the well-known series of mathematical tracts to which he contributed two volumes. His main interests were electrodynamical theory and the application of the theory of conformal mapping to problems in physics.

In his recent presidential address to the Institute of Physics, Sir J. J. THOMSON gave an account of some of his impressions during his recent visit to America. Of interest to teachers of mathematics is his statement that American Universities do not seem designed to produce a large number of men qualified to take up advanced research work. Few of the science students, he says, have the necessary equipment in mathematics, while the stern training which a good student in a great English University has to go through appears to be unknown. In this regard, Great Britain has a distinct advantage which is sorely needed if America is to hold its own in competition. In these remarks the distinguished physicist and mathematician has undoubtedly pointed out one of the weakest points in our current mathematical instruction: the failure to impress sufficiently upon the first rate student of science the absolute necessity of advanced mathematical training.

SUBSCRIPTIONS TO FORTSCHRITTE AND JAHRESBERICHT

Members of the ASSOCIATION are already aware of the extortionate prices demanded for *Jahrbuch über die Fortschritte der Mathematik*, vol. 45 ($19.20), and *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 31, 1922 ($14.00). As the result of extensive correspondence, Professors Bieberbach and Lichtenstein have communicated the following offer to members of the ASSOCIATION: Volume 46 of the *Fortschritte* will be supplied at 25 per cent. discount on the published price, postage and packing being extra. The first Heft of this volume (550 pages, 1923), published at $5.45, therefore costs $4.20. The *Jahresbericht*, vol. 32, 1923, will be sent for $2.25 which includes the dues for the Vereinigung.

In case any *Library*, not in a College or University already an institutional member of the ASSOCIATION, wishes to avail itself of the above rate, it may do so by joining the Vereinigung for which the annual fee is, for the time being, not more than fifty cents per year.

Orders under the above plan should be sent to Professor Dr. L. Bieberbach, Marienbaderstr. 9, Berlin-Schmargendorf, Germany. Payment should be made in dollars, when ordering, by draft either on New York or Berlin.

We are informed that if a considerable number of new subscribers results from these very reasonable rates, a still further reduction in the price per page of the *Fortschritte* may be expected. The number of subscribers now is 130 less than before the war.

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Grad. R. R. 3

FEB 6 1924

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